Mathematics

AIMS Mathematics, 8(8): 18021–18039.
DOI: 10.3934/math.2023916
Received: 10 February 2023
Revised: 28 April 2023
Accepted: 09 May 2023
Published: 25 May 2023

http://www.aimspress.com/journal/Math

Research article

Aczel-Alsina-based aggregation operators for intuitionistic hesitant fuzzy set environment and their application to multiple attribute decision-making process

Wajid Ali[1], Tanzeela Shaheen[1], Iftikhar Ul Haq[1], Hamza Toor[2], Faraz Akram[2], Harish Garg[3,4,*], Md. Zia Uddin[5] and Mohammad Mehedi Hassan[6]

[1] Department of Mathematics, Air University, E-9, Islamabad, Pakistan
[2] Biomedical Engineering Department, Riphah International University, Islamabad, Pakistan
[3] School of Mathematics, Thapar Institute of Engineering & Technology (Deemed University), Patiala-147004, Punjab, India
[4] Department of Mathematics, Graphic Era Deemed to be University, Dehradun, 248002, Uttarakhand, India
[5] Software and Service Innovation, SINTEF Digital, Oslo-0373, Norway
[6] Information Systems Department, College of Computer and Information Sciences, King Saud University, Riyadh 11543, Saudi Arabia

* **Correspondence:** Email: harishg58iitr@gmail.com; Tel: +918699031147.

Abstract: An intuitionistic hesitant fuzzy set is an extension of the fuzzy set which deals with uncertain information and vague environments. Multiple-attribute decision-making problems (MADM) are one of the emerging topics and an aggregation operator plays a vital role in the aggregate of different preferences to a single number. The Aczel-Alsina norm operations are significant terms that handle the impreciseness and undetermined data. In this paper, we build some novel aggregation operators for the different pairs of the intuitionistic hesitant fuzzy sets (IHFSs), namely as Aczel-Alsina average and geometric operators. Several characteristics of the proposed operators are also described in detail. Based on these operators, a multi-attribute decision-making algorithm is stated to solve the decision-making problems. A numerical example has been taken to display and validate the approach. A feasibility and comparative analysis with existing studies are performed to show its superiority.

Keywords: intuitionistic fuzzy sets; IHFSs; Aczel-Alsina aggregation operators; MADM
Mathematics Subject Classification: 03E72, 08A72, 54A40

1. Introduction

Modern decision science considers multiple-attribute decision-making (MADM) to be a crucial topic of research that can decide the appropriate options in accordance with numerous salient qualities [1,2]. When faced with traditional MADM problems, decision-makers (DMs) typically apply clear figures to convey their preferences for the alternative. However, due to a lack of information, a lack of resources, a lack of time, and a lack of quality values, many subjective attribute values are more easily expressed using fuzzy data than by using actual numbers. The theory of fuzzy sets (FSs) was created by Zadeh [3], which many investigators later extended according to the need [4–6]. For every fuzzy set, there exists a set of components and their corresponding membership functions, which assigns a degree of membership to each component in the range of [0,1]. Atanassov [7] offered an intuitionistic fuzzy set (IFS) in 1986. The use of membership and non-membership grades in an IFS allows for the representation of ambiguous and complex information, subject to the constraint that the sum of both grades cannot surpass 1. Another parallel methodology to cope with vagueness was made by Torra [8], who defined a hesitant fuzzy set (HFS). An HFS permits the membership grade occupying a set of possible results of the interval from 0 to 1. HFS is an extended structure of FS that finds a broad application in various complex scenarios. Several scholars have conducted a thorough investigation into the procedures for accumulating HF data and their impact on decision making [9,10].

Mahmood et al. [11] proposed the concept of intuitionistic hesitant fuzzy sets (IHFS) involves the combination of IFS and HFS, where the resulting grades are expressed as a collection of potential results ranging from 0 to 1. Certainly, an IHFS has been established as a powerful instrument for clarifying the fuzziness of the DM difficulties. To achieve this kind of point, Yager [12,13] founded a power average (PA) operator and executes it to MADM problems. Zhang et al. [14] presented Heronian mean aggregation operators for generalization of FSs. Xu et al. [15] introduced several new geometric aggregation operators for IFSs. Senapati et al. [16] described an MADM approach for intuitionistic fuzzy set information. Ayub [17] expanded the Bonferroni mean aggregation for dual hesitant circumstances. Hadi et al. [18] described the Hamacher mean operators to find the best selection during DM.

Triangular norms $(T.\mathcal{N})$ play a vital role during decision-making. The notion of $T.\mathcal{N}$ was first introduced in the supposition of probabilistic metric spaces by Menger [19]. Drosses [20] presented some generalized t-norms structures. Descharijver [21] used the abovementioned notion on the intuitionistic fuzzy environment. Boixader [22] also investigated some t-norms and t-conorms during his research. Similarly, a few scholars have investigated this area deeply [23–25]. A concept of triangular norms [26–28] have extensively reviewed recent well-organized research on the qualities and related elements of $T.\mathcal{N}s$. Aczel and Alsina [29] introduced new procedures in 1982 under the names Aczel Alsina $T.\mathcal{N}$ and Aczel-Alsina $T.C\mathcal{N}$, which prioritize changeability with parameter activity. Ye et al. [30] introduced Aczel-Alsina operators for Z-Numbers and applied in MADM. In the literature, some approaches related to MADM problems, we refer you to read the articles [16,31–36].

It is observed from the above literature that several algorithms are addressed by the various researchers to handle MADM problems. However, in this existing literature, it is found that they have considered that all the attributes are independent to each other. However, in day-today life problems, one parameter may influence others and thus, it is necessary to consider the information during the analysis. Another feature obtained from the review is that during the information collection phase, an expert may provide more than one decision on a single information. Thus, the model of the intuitionistic hesitant fuzzy set plays a vital role. Furthermore, the IHFS is a generalization of the existing theories. It is vital to convey the shaky facts in a much more beneficial way so that the best option(s) for the MADM concerns may be selected. It is critical to cope with how to take the relationship between input arguments into consideration as well. From this inspiration, we combine two novel frameworks Intuitionistic hesitant fuzzy sets and Aczel-Alsina aggregation operators. Based on the aggregation operators and under the data of intuitionistic hesitant fuzzy, the multiple attributes decision making techniques is investigated.

The main impact of this article is described as below:

1) Consider the environment of IHFS to handle the uncertainties in the data. In this set, set of values are considered in terms of membership and non-membership values.
2) Utilizing the feature of the Aczel-Alsina norm operators, we define several weighted aggregation operators, namely intuitionistic hesitant fuzzy weighted averaging and geometric operators. Additionally, we stated their fundamental properties.
3) To design a novel MADM algorithm based on the defined operators.
4) To produce a numerical example to display the applicability of the stated algorithm and compare their results with existing studies.

The remaining parts of the article are arranged below. Section 2 delivers a short overview of the basic concepts. In Section 3, we state the series of Aczel-Alsina aggregation operation rules for the IHFNs such as the $IHF\mathcal{A}_{\mathcal{A}}WA_{\delta}$ operator, the $IHF\mathcal{A}_{\mathcal{A}}OWA_{\delta}$ operator, and the $IHF\mathcal{A}_{\mathcal{A}}WA_{\delta}$ operator, and their effective attributes. Section 4 presents a multiple-attribute decision making (MADM) algorithm that utilizes IHF data and the $IHFA_{A}WA_{\delta}$ operator to represent characteristic values. In Section 5, an example is given to establish the use of the proposed model for selecting a gadget. Section 6 concludes the paper.

2. Preliminaries

This section covers the fundamental models of IHFSs and several ideas associated with Aczel-Alsina $T.\mathcal{N}s$, $T.\mathcal{CN}$, and aggregation operators. The most commonly used abbreviations in the paper are summarized in Table 1.

Table 1. Symbols with description.

Symbols	Description	Symbols	Description
FS	Fuzzy Set	$T.\mathcal{N}s$	Triangular Norms
IHFS	Intuitionistic Hesitant Fuzzy Sets	$T.\mathcal{CN}$	Triangular Co-Norms
MADM	Multiple-attribute Decision Making	IHFPWA	Intuitionistic hesitant fuzzy power weighted average
MG	Membership Grades	$IHF\mathcal{A}_{\mathcal{A}}A$	Intuitionistic hesitant fuzzy Aczel-Alsina average
NMG	Non-membership Grades	$IHF\mathcal{A}_{\mathcal{A}}WA$	Intuitionistic hesitant fuzzy Aczel-Alsina weighted average
DM	Decision-maker	$IHF\mathcal{A}_{\mathcal{A}}OWA_{\delta}$	Intuitionistic hesitant fuzzy Aczel-Alsina ordered weighted average
$\boldsymbol{Scr}$	Score Function	$\mathcal{H}ac$	Accuracy Function

2.1. An overview of intuitionistic hesitant fuzzy sets

Atanassov [7] suggested the idea of IFS as a development of FS. While FS gives the membership grade of an element within a specific collection in the range of [0, 1], IFS supplies both the membership grade (MG) and non-membership grade (NMG) instantaneously.

Definition 1. [7] The IFS H over the universe U is represented by a pair of mappings, $m(s)$ and $n(s)$, which can be mathematically expressed using the following form:

$$H = \langle s, m_H(s), n_H(s)) | s \in U \rangle. \tag{1}$$

The functions $m_H(s)$ and $n_H(s)$ denote the MG and NMG, respectively, for a given $s \in U$, subject to the condition that their sum is between 0 and 1 (i.e., $0 \leq m(s) + n(s) \leq 1$).

For any IFS H defined over U, the indeterminacy grade of an element e with respect to H is denoted as $p_H(s)$ and is defined as $p_H(s) = 1 - m_H(s) - n_H(s), \forall\, s \in U$.

Mahmood et al. [11] proposed the combination of IFS with HFS results in a more generalized form, identified as IHFSs. In IHFSs, both the membership grade and non-membership grade denotes a set of values ranging from 0 to 1. The basic definition and operations are presented as follow:

Definition 2. [11] An IHFS H defined over U is represented by a pair of mappings, $m(s)$ and $n(s)$, which can be mathematically expressed using the following form:

$$H = \langle s, m_H(s), n_H(s)) | s \in U \rangle \tag{2}$$

The mappings $m_H(s)$ and $n_H(s)$ represent a set of possible membership grades (MGs) and non-membership grades of the elements s $\in$ U to the group H, where the values are between 0 and 1. The condition that $0 \leq max(m_H(e)) + max(n_H(e)) \leq 1$ is also satisfied. For the sake of convenience, $(m(e), n(e))$ is commonly referred to as an IHFN throughout the study.

Definition 3. The functions for "score $Scr\,(H)$ and accuracy" $Hac(H)$ are designed and symbolized for any IHFNs $H = (m_H, n_H)$ as follows:

$$Scr(H) = \frac{S(m_H) - S(n_H)}{2}, Scr(H) \in [-1,1] \tag{3}$$

$$Hac(H) = \frac{S(m_H) + S(n_H)}{2}, Hac(H) \in [0,1]. \tag{4}$$

Where, $S(m_H) = \frac{sum\ of\ all\ elements\ in\ (m_H)}{order\ of\ (m_H)}$, $S(n_H) = \frac{sum\ of\ all\ elements\ in\ (n_H)}{order\ of\ (n_H)}$.

Definition 4. [11] Let $H_1 = (m_1, n_1)$ and $H_2 = (m_2, n_2)$ be IHFSs, and the basic operations are defined as below:

(i) $H_1 \oplus H_2 = \bigcup_{\substack{a_1 \varepsilon m_1 \\ a_2 \varepsilon m_2 \\ b_1 \varepsilon n_1 \\ b_2 \varepsilon n_2}} (\{a_1 + a_2 - a_1 a_2\}, \{b_1 b_2\})$

(ii) $H_1 \otimes H_2 = \bigcup_{\substack{a_1 \varepsilon m_1 \\ a_2 \varepsilon m_2 \\ b_1 \varepsilon n_1 \\ b_2 \varepsilon n_2}} (\{a_1 a_2\}, \{b_1 + b_2 - b_1 b_2\})$

(iii) $\lambda H_1 = \bigcup_{\substack{a \varepsilon m_1 \\ b \varepsilon n_1}} (\{1 - (1-a)^\lambda\}, \{b^\lambda\}), \lambda > 0$

(iv)$H_1^{\lambda} = \bigcup_{\substack{a \varepsilon m_1 \\ b \varepsilon n_1}} \left(\{(a)^{\lambda}\}, \{1-(1-b)^{\lambda}\}\right), \lambda > 0$

(v) $H_1^c = \left(\{b_{n_1}\}, \{a_{m_1}\}\right)$.

Definition 5. Consider a set of IHFSs represented as $H_j = (m_j, n_j)$, and let $\delta_j = (\delta_1, \delta_2, \ldots, \delta_n)^T$ denote the weights for H_j, where $\sum_{j=1}^{n} \delta_j = 1$. "The IHFPWA operator is a mapping IHFPWA": H^n→W such that:

$$IHFPHA_{\delta}(H_1, H_2, \ldots, H_n) = \frac{\bigoplus_{j=1}^{n} (\delta_j (1 + T(H_j) H_j)}{\sum_{j=1}^{n} \delta_j (1 + T(H_j))}$$

$$= \bigcup_{\substack{a_j \varepsilon m_j \\ b_j \varepsilon n_j}} \left(1 - \prod_{j=1}^{n} (1 - (a_j)^{\frac{\delta_j((1+T(H_j))}{\sum_{j=1}^{n} \omega_j (1+T(H_j))}}, \ \prod_{j=1}^{n} (b_j)^{\frac{\delta_j (1+T(H_j))}{\sum_{j=1}^{n} \delta_j (1+T(H_j))}} \right),$$

where

$$T(H_j) = \bigcup_{\substack{m_j \varepsilon H_j \\ n_j \varepsilon H_j}} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \delta_j Sup(H_j, H_i) \right).$$

Definition 6. Let $H_j = (m_j, n_j)$ denote a set of IHFSs with their corresponding weights $\delta_j = (\delta_1, \delta_2, \ldots, \delta_n)^T$ where $\delta_j > 0$ and $\sum_{j=1}^{n} \delta_j = 1$. The IHFPOWA operator is a mapping IHFPOWA: $H^n \to H$, described as:

$$IHFPOHA_{\delta}(H_1, H_2, \ldots, H_n) = \frac{\bigoplus_{j=1}^{n} (\delta_j (1 + T(H_{\sigma(j)}) H_{\sigma(j)})}{\sum_{j=1}^{n} \delta_j (1 + T(H_{\sigma(j)}))}$$

$$= \bigcup_{\substack{a_{\sigma(j)} \varepsilon m_j \\ b_{\sigma(j)} \varepsilon n_j}} \left(1 - \prod_{j=1}^{n} (1 - (a_{\sigma(j)})^{\frac{(\delta_j (1+T(H_{\sigma(j)}))}{\sum_{j=1}^{n} \delta_j (1+T(H_{\sigma(j)}))}}, \ \prod_{j=1}^{n} (b_{\sigma(j)})^{\frac{(\delta_j (1+T(H_{\sigma(j)}))}{\sum_{j=1}^{n} \delta_j (1+T(H_{\sigma(j)}))}} \right).$$

2.2. An overview of Aczel-Alsina operators

The specific class of functions known as triangular norms ($T.\mathcal{N}s$) can be used to interpret the intersection of fuzzy logic and FSs. Menger [19] created an idea of $T.\mathcal{N}s$. The concepts that are essential for the development of this article are widely used in various applications related to data aggregation and decision-making. In the following sections, we will discuss these key concepts in detail.

Definition 7. A function C: [0, 1]×[0, 1] → [0, 1] is a $T.\mathcal{N}s$ is fulfilled following characters, $\forall\ e, f, g \in$ [0, 1],

(i) Symmetry: $C\ (e,\ f) = C\ (f,\ e)$.

(ii) Associativity: $C\ (e,\ C\ (f,\ g))$ e= $C\ (C\ (e,\ f),\ g)$.

(iii) Monotonicity: $C\ (e,\ f) \leq C\ (e,\ g)$ if $f\ \leq g$

(iv)One Identity: $C\ (1,\ e) = e$.

Examples of $T.\mathcal{N}s$ are: $\forall\ e, f, g \in$ [0, 1],

(i) Product $T.\mathcal{N}$: C_{pro} $(e,\ f) = e.\ f$;
(ii) Minimum $T.\mathcal{N}$: C_{min} $(e,\ f) = min\,(e,\ f)$.
(iii) Lukasiewicz $T.\mathcal{N}$: C_{luk} $(e,\ f) = max\,(e+f-1, 0)$.
(iv)Drastic $T.\mathcal{N}$:

$$\mathrm{C}_{dra}\ (e,\ f) = \begin{cases} e, & if\ f = 1 \\ f, & if\ e = 1. \\ 0, & otherwise \end{cases}$$

Definition 8. The function D: [0, 1]×[0, 1] → [0, 1] is $T.C\mathcal{N}s$ if the following properties are convinced: $\forall\ e, f, g \in$ [0, 1],
(i) Symmetry: $D\ (e,\ f) = D\ (f\ ,\ e)$.
(ii) Associativity: $D\ (e,\ D\ (f\ ,\ g))\ e = D\ (D\ (e\ ,\ f),\ g)$.
(iii) Monotonicity: $D\ (e,\ f) \leq D\ (e,\ g)$ if $f\ \leq g$
(iv)Zero Identity: $D\ (0,\ e) = e$;
Examples of $T.C\mathcal{N}s$ are: $\forall\ e, f, g \in$ [0, 1],
(i) Probabilistic sum $T.C\mathcal{N}$: $D_{\mathrm{PS}}\ (e,\ f) = e + f - e, f$;
(ii) Maximum $T.C\mathcal{N}$: $D_{max}\ (e,\ f) = max\,(e,\ f)$.
(iii) Lukasiewicz $T.C\mathcal{N}$: $D_{\mathrm{luk}}\ (e,\ f) = min\,\{e\ +\ f, 1\}$.
(iv)Drastic $T.C\mathcal{N}$:

$$D_{dra}\ (e,\ f) = \begin{cases} e, & if\ f = 0 \\ f, & if\ e = 0. \\ 1, & otherwise \end{cases}$$

Definition 9. [29] Aczel-Alsina offered a unique triangular norms and triangular co-norms described as:

$$C_{\breve{\mathrm{A}}}^{\varphi}(e, f) = \begin{cases} C_{dra}(e, f), & if\ \varphi = 0 \\ min(e, f), & if\ \varphi = \infty \\ e^{-((-\log l)^{\varphi} + (-\log m)^{\varphi})^{1/\varphi}}, & otherwise \end{cases}$$

and

$$D_{\breve{\mathrm{A}}}^{\varphi}(e, f) = \begin{cases} D_{dra}(e, f), & if\ \varphi = 0 \\ max(e, f) & if\ \varphi = \infty \\ 1 - e^{-((-\log(1-l))^{\varphi} + (-\log(1-m))^{\varphi})^{1/\varphi}}, & otherwise \end{cases}.$$

3. Aczel-Alsina operators for intuitionistic hesitant fuzzy sets

In this section, we will explore the $\mathcal{A}_{\mathcal{A}}$ operations for IHFSs and examine the major properties of these mappings.

The Aczel-Alsina $T.\mathcal{N}$ C and $T.C\mathcal{N}$ D are used to define the product $C_{\breve{\mathrm{A}}}$ and sum $D_{\breve{\mathrm{A}}}$ operations for IHFSs H_1 and H_2 as follows:

$$H_1 \otimes H_2 = \left\{< s, C_{\breve{\mathrm{A}}}\{m_{H_1}(s), m_{H_2}(s)\}, D_{\breve{\mathrm{A}}}\{n_{H_1}(s), n_{H_2}(s)\} >: s \in U\right\}$$

$$H_1 \oplus H_2 = \{< s, D_{\check{A}}\{m_{H_1}(s), m_{H_2}(s)\}, C_{\check{A}}\{n_{H_1}(s), n_{H_2}(s)\} >: s \in U\}.$$

Definition 10. Consider $H_1 = (m_1, n_1)$ and $H_2 = (m_2, n_2)$ be two IHFSs, where $a_u, b_u \in m_1$ and $a_u, b_u \in n_2$ such that $u = 1,2 \ldots, p'$ with $ℶ \geq 1$ and $\tau > 0$. Let $\rho_j = \left(\frac{1}{p' \sum_{u=1}^{p'} a_u(m_j)}\right)$ and $\phi_j = \left(\frac{1}{p' \sum_{u=1}^{p'} b_u(n_j)}\right)$ be the membership grade and non-membership grade for IHFNs for Aczel-Alsina aggregation operators. The Aczel-Alsina operations for intuitionistic hesitant fuzzy numbers (IHFNs) can be described as follows:

(i) $H_1 \oplus H_2 = < 1 - e^{-\left((-\log(1-\rho_1))^{ℶ} + (-log(1-\rho_2))^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left((-log(\phi_1))^{ℶ} + (-log(\phi_2))^{ℶ}\right)^{\frac{1}{ℶ}}} >$

(ii) $H_1 \otimes H_2 = \langle e^{-\left((-log(\rho_1))^{ℶ} + (-log(\rho_2))^{ℶ}\right)^{\frac{1}{ℶ}}}, 1 - e^{-\left(\left(-log(1-(\phi_1))\right)^{ℶ} + \left(-log(1-(\phi_2))\right)^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

(iii) $\tau \mathrm{H} = \langle 1 - e^{-\left(\tau \left(-log(1-(\rho))\right)^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left(\tau (-log(\phi))^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

(iv) $\mathrm{H}^{\tau} = \langle e^{-\left(\tau (-log(\rho))^{ℶ}\right)^{\frac{1}{ℶ}}},\ 1 - e^{-\left(\tau \left(-log(1-(\phi))\right)^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle.$

Theorem 1. For two IHFNs $H_1 = (m_{W_1}, n_{W_1})$ and $H_2 = (m_{W_2}, n_{W_2})$, with $ℶ \geq 1$, $\tau > 0$. We have

(i) $H_1 \oplus H_2 = H_2 \oplus H_1$
(ii) $H_1 \otimes H_2 = H_2 \otimes H_1$
(iii) $\tau(H_1 \oplus H_2) = \tau H_1 \oplus \tau H_2$
(iv) $(H_1 \otimes H_2)^{\tau} = H_1^{\tau} \otimes H_2^{\tau}$
(v) $H^{\tau_1} \otimes H^{\tau_2} = H^{(\tau_1 + \tau_2)}$.

Proof. For three IHFNs H, H_1, H_2 and for $\tau, \tau_1, \tau_2 > 0$, as defined in Definition 10, the following relations hold:

(i) $H_1 \oplus H_2$

$= \langle 1 - e^{-\left(\left(-\log(1-(\rho_1))\right)^{ℶ} + \left(-log(1-(\rho_2))\right)^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left((-log(\phi_1))^{ℶ} + (-log(\phi_2))^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

$= \langle 1 - e^{-\left(\left(-\log(1-(\rho_1))\right)^{ℶ} + \left(-log(1-(\rho_1))\right)^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left((-log(\phi_2))^{ℶ} + (-log(\phi_1))^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

$= H_2 \oplus H_1.$

(ii) It is straightforward.

(iii) Let $f = 1 - e^{-\left(\left(-\log(1-(\rho_2))\right)^{ℶ} + \left(-log(1-(\rho_1))\right)^{ℶ}\right)^{\frac{1}{ℶ}}}$ then $log(1-f) = -\left(\left(-\log(1-(\rho_2))\right)^{ℶ} + \left(-log(1-(\rho_1))\right)^{ℶ}\right)^{\frac{1}{ℶ}}.$

Using this, we get

$\tau(H_1 \oplus H_2)$

$= \tau \langle 1 - e^{-\left(\left(-\log(1-(\rho_1))\right)^{ℶ} + \left(-log(1-(\rho_2))\right)^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left((-log(\phi_1))^{ℶ} + (-log(\phi_2))^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

$= \langle 1 - e^{-\left(\tau\left(-\log(1-(\rho_1))\right)^{ℶ} + \left(-log(1-(\rho_2))\right)^{ℶ}\right)^{\frac{1}{ℶ}}},\ e^{-\left(\tau(-log(\phi_1))^{ℶ} + (-log(\phi_2))^{ℶ}\right)^{\frac{1}{ℶ}}} \rangle$

$$= \langle 1 - e^{-\left(\tau\left(-\log(1-(\rho_1))\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ e^{-\left(\tau\left(-log(\phi_1)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle \oplus$$

$$\langle 1 - e^{-\left(\tau\left(-log(1-(\rho_2))\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ e^{-\left(\tau\left(-log(\phi_2)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= \tau H_1 \oplus \tau H_2.$$

(iv) $\tau_1 H \oplus \tau_2 H = \langle 1 - e^{-\left(\mathrm{T}_1\left(-log(1-(\rho))\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ e^{-\left(\tau_1\left(-log(\phi)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$

$$\oplus \langle 1 - e^{-\left(\tau_1\left(-log(1-(\rho))\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ e^{-\left(\tau_1\left(-log(\phi)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= \langle 1 - e^{-\left((\tau_1+\mathrm{T}_2)\left(-log(1-(\rho))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}, e^{-\left((\tau_1+\mathrm{T}_2)\left(-log(\phi)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= (\tau_1 + \tau_2)H.$$

(v) $(H_1 \otimes H_2)^{\tau}$

$$= \langle e^{-\left(\left(-log(\rho_1)\right)^{\daleth} + \left(-log(\rho_2)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}, 1 - e^{-\left(\left(-log(1-(\phi_1))\right)^{\daleth} + \left(-log(1-(\phi_2))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle^{\mathrm{T}}$$

$$= \langle e^{-\left(\tau\left(-log(\rho_1)\right)^{\daleth} + \left(-log(\rho_2)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}, 1 - e^{-\left(\tau\left(-log(1-(\phi_1))\right)^{\daleth} + \left(-log(1-(\phi_2))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= \langle e^{-\left(\tau\left(-log(\rho_1)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}, 1 - e^{-\left(\tau\left(-log(1-(\phi_1))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$\otimes \langle e^{-\left(\tau\left(-log(\rho_2)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}, 1 - e^{-\left(\mathrm{T}\left(-log(1-(\phi_2))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= H_1^{\tau} \otimes H_2^{\tau}.$$

(vi) $H^{\tau_1} \otimes H^{\tau_2}$

$$= \langle e^{-\left(\mathrm{T}_1\left(-log(\rho)\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ 1 - e^{-\left(\tau_1\left(-log(1-(\phi))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle \otimes \langle e^{-\left(\tau_2\left(-log(\rho)\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ 1 - e^{-\left(\mathrm{T}_2\left(-log(1-(\phi))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= \langle e^{-\left((\tau_1+\tau_2)\left(-log(\rho)\right)^{\daleth}\right)^{\frac{1}{\daleth}}},\ 1 - e^{-\left((\tau_1+\tau_2)\left(-log(1-(\phi))\right)^{\daleth}\right)^{\frac{1}{\daleth}}}\rangle$$

$$= H^{(\tau_1+\tau_2)}.$$

Intuitionistic hesitant fuzzy Aczel-Alsine average aggregation operators

We will now present several "average aggregation operators using the Aczel-Alsina operations".

Definition 11. For a collection of IHFNs, denoted by $H_i = \left(m_{H_i}, n_{H_i}\right), \forall i \in N,$ the weight vector $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is defined for these IHFNs, where $\delta_i > 0, \delta_i \in [0,1]$ and $\sum_{i=1}^{n} \delta_i = 1$. The $IHFA_A\ HA$ operator is a mapping IHF$A_A\ HA$: $H^n \to H$, which is designed as below:

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2 \ldots, H_n) = \oplus_{i=1}^{n}(\delta_i H_i) = \delta_1 H_1 \oplus \delta_2 H_2 \oplus, \ldots, \oplus\, \delta_n H_n.$$

The following theorem can be derived from Definition 11 for IHFNs.

Theorem 2. Consider we have a gathering of IHFNs, $H_i = \left(m_{H_i}, n_{H_i}\right),\ where\ i \in N$, with assigned weights δ. When the $IHFA_A\ HA_{\delta}$ operator is applied to these IHFNs, the obtained result is also an IHFN.

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \ldots, H_n) = \oplus_{i=1}^{n}(\delta_i H_i) = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{n}\delta_i(-\log(\rho_i))\right)^{1/\daleth}},\ \mathrm{e}^{-\left(\sum_{i=1}^{n}\delta_i(-\log(\phi_i))^{\daleth}\right)^{1/\daleth}}\rangle.$$

(5)

Proof. The theorem can be proven using a mathematical induction as follows:
(I) Consider $i = 2$, we get

$$\delta_1 H_1 = \langle 1 - \mathrm{e}^{-\left(\delta_1(-\log(1-\rho_1)))\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_1(-\log(\phi_1))^{\beth}\right)^{1/\beth}}\rangle$$

$$\delta_2 H_2 = \langle 1 - \mathrm{e}^{-\left(\delta_2(-\log(1-\rho_2)))\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_2(-\log(\phi_2))^{\beth}\right)^{1/\beth}}\rangle.$$

Using Definition 10, we can derive the following:

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2) = \delta_1 H_1 \oplus \delta_2 H_2$$

$$= \langle 1 - \mathrm{e}^{-\left(\delta_1(-\log(1-\rho_1))^{\beth}\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_1(-\log(\phi_1))^{\beth}\right)^{1/\beth}}\rangle \oplus \langle 1 - \mathrm{e}^{-\left(\delta_2(-\log(1-\rho_2))^{\beth}\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_2(-\log(\phi_2))^{\beth}\right)^{1/\beth}}\rangle$$

$$= \langle 1 - \mathrm{e}^{-\left(\delta_1(-\log(1-\rho_1))^{\beth}+\delta_2(-\log(1-\rho_2))^{\beth}\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_1(-\log(\phi_1))^{\beth}+\delta_2(-\log(\phi_2))^{\beth}\right)^{1/\beth}}\rangle$$

$$= \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{2}\delta_i(-\log(1-\rho_i))^{\beth}\right)^{1/\beth}},\ \mathrm{e}^{-\left(\sum_{i=1}^{2}\delta_i(-\log(\phi_i))^{\beth}\right)^{1/\beth}}\rangle.$$

Therefore, Eq (5) is fulfilled for $i = 2$.
(II) Assume that for $i = k$, Eq (5) subsequently fulfills, and the following expression is obtained.

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \dots, H_k) = \oplus_{i=1}^{k}(\delta_i H_i) = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{k}\delta_i\left(-\log(1-\rho_i)^{\beth}\right)\right)^{1/\beth}},\ \mathrm{e}^{-\left(\sum_{i=1}^{k}\delta_i(-\log(\phi_i))^{\beth}\right)^{1/\beth}}\rangle.$$

Now, considering the case of $i = k + 1$, we obtain:

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \dots, H_k, H_{k+1}) = \oplus_{s=1}^{k}(\delta_i H_i) \oplus (\delta_{k+1} H_{k+1}) = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{k}\delta_i(-\log(1-\rho_i)^{\beth})\right)^{1/\beth}},\ \mathrm{e}^{-\left(\sum_{i=1}^{k}\delta_i(-\log(\phi_i))^{\beth}\right)^{1/\beth}}\rangle \oplus \langle 1 - \mathrm{e}^{-\left(\delta_{k+1}(-\log(1-\rho_{k+1})^{\beth})\right)^{1/\beth}},\ \mathrm{e}^{-\left(\delta_{k+1}(-\log(\phi_{k+1}))^{\beth}\right)^{1/\beth}}\rangle = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{k+1}\delta_i(-\log(1-\rho_i)^{\beth})\right)^{1/\beth}},\ \mathrm{e}^{-\left(\sum_{i=1}^{k+1}\delta_i(-\log(\phi_i))^{\beth}\right)^{1/\beth}}\rangle.$$

Therefore, we have shown that Eq (6) is valid for $i = k + 1$, assuming that it is valid for $i = k$.

From (I) and (II), it can be concluded that Eq (6) holds for all values of i.

Using the $IHF\mathcal{A}_{\mathcal{A}}HA$ operator, we were able to effectively demonstrate the relevant characteristics.

Property 1. (Idempotency). If $H_i = (m_{H_i}, \nu_{H_i})$ for all $i \in N$ are equal, that is, $H_i = H$, then applying $IHFA_AHA_\delta$ operator on $H_1, H_2, \dots, H_i$ results in H.

Property 2. (Boundedness). If a set of IHFNs, $H_i = (m_{H_i}, n_{H_i})$, is given, where i=1,2,...,n, then let $H^- = min\,(H_1 H_2 \dots, H_n)$ and $H^+ = max\,(H_1 H_2 \dots, H_n)$. Then, it follows that:
$H^- \leq IHFA_AHA\delta\ (H_1\ H_{2,} \dots, H_n\ ) \leq H^+$.

Property 3. (Monotonicity). For H_i and H_i' be two IHFNs. Let $H_i \leq H_i'$ for all i hen $IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}\left(H_{1,}H_{2,} \dots, H_n\right) \leq IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}\left(H_{1,}'H_{2,}' \dots, H_n'\right)$.

We will now introduce IHF Aczel-Alsina ordered weighted averaging $(IHFA_AOHA_\delta)$ operations.

Definition 12. Consider we have a collection of IHFNs $H_i = (m_{H_i}, \nu_{H_i}), (i = 1,2, \dots, n)$, and weights assigned to each IHFN, $\delta = (\delta_1, \delta_2, \dots, \delta n\ )^T$ and $\delta_i \in [0,1]$. The $IHFA_AOHA_\delta$ operator can be defined as a function: $IHFA_AOHA_\delta: H^n \to H$.

$$\text{IHF-}\mathcal{A}_{\mathcal{A}}\text{-WA}(H_1, H_2 \dots, H_n) = \oplus_{s=1}^{n}\left(\delta_i H_{\sigma(i)}\right)$$
$$= \delta_1 H_{\sigma(1)} \oplus \delta_2 H_{\sigma(2)} \oplus, \dots, \oplus \delta_n H_{\sigma(n)}.$$

Where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ are the permutations of $\forall i \in N$, containing $H_{\sigma(n-1)} \geq H_{\sigma(n)}, \forall i$.
The following result was obtained from Definition 12.
Theorem 2. The result of applying the $IHFA_AOHA$ operator on an accumulation of IHFNs $H_i = (m_{H_i}, \nu_{H_i}), (i = 1,2,\dots,n)$ with assigned weights δ and $\sum_{i=1}^{n} \delta_i = 1$ is also an IHFN.

$$IHF\mathcal{A}_{\mathcal{A}}OHA_\delta(H_1, H_2, \dots, H_n) = \oplus_{s=1}^{n}\left(\delta_i H_{\sigma(i)}\right) = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{n} \delta_i (-\log(1-\rho_{\sigma(i)}))\right)^{1/\beth}}, \; \mathrm{e}^{-\left(\sum_{i=1}^{n} \delta_i \left(-\log(\phi_{\sigma(i)})\right)^{\beth}\right)^{1/\beth}} \rangle. \tag{6}$$

Where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ are the permutations of every i, containing $H_{\sigma(n-1)} \geq H_{\sigma(n)}$.
The properties related to $IHFA_AOHA_\delta$ operator can be verified by utilizing it.
Property 4. If $H_i = (m_{H_i}, \nu_{H_i})$ for all $i \in N$ are equal, that is, $H_i = H$, then applying $IHFA_AOHA_\delta$ operator on $H_1, H_2, \dots, H_i$ results in H.
Property 5. If a set of IHFNs, $H_i = (m_{H_i}, n_{H_i})$, is given, where i=1,2,...,n, then let $H^- = min\,(H_1 H_2 \dots, H_n)$ and $H^+ = max\,(H_1 H_2 \dots, H_n)$. Then, it follows that:
$H^- \leq IHFA_AOHA\,(H_1\, H_{2,} \dots, H_n\,) \leq H^+$.
Property 6. For H_i and H_i' be two IHFNs. Let $H_i \leq H_i'$ for all i then $IHF\mathcal{A}_{\mathcal{A}}OHA_\delta\left(H_{1,} H_{2,} \dots, H_n\right) \leq IHF\mathcal{A}_{\mathcal{A}}HA_\delta\left(H_{1,}' H_{2,}' \dots, H_n'\right)$.
Property 7. Let H_i and H_i' be two sets of IHFNs, then $IHF\mathcal{A}_{\mathcal{A}}OHA_\delta\left(H_{1,} H_{2,} \dots, H_n\right) = IHF\mathcal{A}_{\mathcal{A}}OHA_\delta\left(H_{1,}' H_{2,}' \dots, H_n'\right)$ where $H_i'(i \in N)$ is any permutation of $H_i\ (i \in N)$.
Below is the definition of a hybrid aggregation operator that can be developed based on Definitions 11 and 12.
Definition 13. Assuming that we have an accumulation of IHFNs denoted by $H_i = (m_{H_i}, n_{H_i})$, a set of assigned weights $\delta = (\delta_1, \delta_2, \dots, \delta_n)^T$ for each H_i, and a new IHFN $\dot{H}_i = n\delta_i H_i$, the $IHFA_AHA_\delta$ operator is defined as a function $IHFA_AHA_\delta: H^n \to H$.

$$IHF\mathcal{A}_{\mathcal{A}}HA_\delta(H_1, H_2 \dots, H_n) = \oplus_{i=1}^{n}\left(\delta_i \dot{H}_{\sigma(i)}\right)$$
$$= \delta_1 \dot{H}_{\sigma(1)} \oplus \delta_2 \dot{H}_{\sigma(2)} \oplus, \dots, \oplus \delta_n \dot{H}_{\sigma(n)}.$$

Where $(\sigma(i))$ signifies the permutation of all i, holding $\dot{H}_{\sigma(n-1)} \geq \dot{H}_{\sigma(n)}$.
Definition 13 leads to the following theorem:
Theorem 3. The application of the $IHFA_A\,HA_\delta$ operator on the IHFNs $H_i = (m_{H_i}, \nu_{H_i})$ yields a result that is also an IHFN.

$$IHF\mathcal{A}_{\mathcal{A}}HA_\delta(H_1, H_2, \dots, H_n) = \oplus_{s=1}^{n}\left(\delta_i \dot{H}_{\sigma(i)}\right) = \langle 1 - \mathrm{e}^{-\left(\sum_{i=1}^{n} \delta_i (-\log(1-\rho_{\sigma(i)}))\right)^{1/\beth}}, \; \mathrm{e}^{-\left(\sum_{i=1}^{n} \delta_i \left(-\log(\phi_{\sigma(i)})\right)^{\beth}\right)^{1/\beth}} \rangle.$$

Proof. Proof is not provided.
Theorem 4. The $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ operators are simplifications of the $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ and $IHF\mathcal{A}_{\mathcal{A}}OHA_\delta$ operators.

Proof.

(1) Let $\delta = \left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)^T$ Then

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \ldots, H_n)$$

$$= \delta_1 \dot{H}_{\sigma(1)} \oplus \delta_2 \dot{H}_{\sigma(2)} \oplus \ldots, \oplus \delta_n \dot{H}_{\sigma(\mathrm{n})}$$

$$= \frac{1}{n}\left(\delta_1 \dot{H}_{\sigma(1)} \oplus \dot{H}_{\sigma(2)} \oplus \ldots, \oplus \dot{H}_{\sigma(n)}\right)$$

$$= \delta_1 H_{\sigma(1)} \oplus \delta_2 H_{\sigma(2)} \oplus \ldots, \oplus \delta_n H_{\sigma(n)}$$

$$= IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \ldots, H_n).$$

(2) Let $\delta = \left(\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)$. Then

$$IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}(H_1, H_2, \ldots, H_n)$$

$$= \delta_1 \dot{H}_{\sigma(1)} \oplus \delta_2 \dot{H}_{\sigma(2)} \oplus \ldots, \oplus \delta_n \dot{H}_{\sigma(\mathrm{n})}$$

$$= \delta_1 H_{\sigma(1)} \oplus \delta_2 H_{\sigma(2)} \oplus \ldots, \oplus \delta_n H_{\sigma(n)}$$

$$= IHF\mathcal{A}_{\mathcal{A}}OHA_{\delta}(H_1, H_2, \ldots, H_n),$$

which completes the proof.

4. MADM algorithm under intuitionistic hesitant fuzzy environment

This part shows the usage of $IHF\mathcal{A}_{\mathcal{A}}$ operators to MADM through intuitionistic hesitant fuzzy data. Suppose $A_i, \forall i$ is distinct groups of alternatives and $G = \{G_1, G_2, \ldots, G_n\}$ is the collection of attributes. The assigned weight $\delta = (\delta_j), \forall j$ for all attributes, where $\sum_{j=1}^{n} \delta_j = 1$. Let the IHF decision matrix $\boldsymbol{\mathcal{R}} = (\mathcal{Y}_{ij})_{m \times n}$ be given to the decision maker, where IHFNs $\mathcal{Y}_{ij} = (\{m_{\mathrm{H}_i j}\}, \{n_{\mathrm{H}_{ij}}\})$ represents alternatives. Therefore, the IHF decision matrix (D.Mat) $\mathcal{R}$ may be stated in the following shape,

$$\mathcal{R} = \begin{array}{c} \cdot \\ \cdot \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_m \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & \cdots & G_n \end{array} \\ \begin{pmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} & \ldots & \mathcal{Y}_{1n} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} & & \mathcal{Y}_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ \mathcal{Y}_{m1} & \mathcal{Y}_{m2} & \cdots & \mathcal{Y}_{mn} \end{pmatrix}, \end{array} \tag{7}$$

where each one of the $\mathcal{Y}_{ij} = (m_{\mathrm{H}_{ij}}, n_{\mathrm{H}_{ij}})$ contributes to IHFN. The preceding procedures must be utilized to set up the MADM method in the IHF information. The $IHF\mathcal{A}_{\mathcal{A}}H\boldsymbol{A}_{\boldsymbol{\delta}}$ operator is used to pick the best alternative. The detailed process is described in the following steps.

Step 1. Convert the IHF decision matrix $\mathcal{R} = (\mathcal{Y}_{ij})_{m \times n}$ into normalization matrix $\bar{\mathcal{R}} = (\bar{\mathcal{Y}}_{ij})_{m \times n}$

$$\bar{\mathcal{Y}}_{ij} = \begin{cases} \mathcal{Y}_{ij} & for\ benifit\ attributes\ G_n, \\ (\mathcal{Y}_{ij})^c & for\ cost\ attributes\ G_n, \end{cases} \tag{8}$$

where $(\mathcal{Y}_{ij})^c$ is the complement of $\mathcal{Y}_{ij}$, so as $(\mathcal{Y}_{ij})^c = (\{n_{\mathrm{H}_i j}\}, \{m_{\mathrm{H}_{ij}}\})$.

Normalization is needed whenever two kinds of attributes (cost attributes and benefit attributes) explains the alternatives otherwise skipped this step.

Step 2. For participants $A_i (i = 1,2, \ldots, m)$ determine all the IHF values $\mathcal{Y}_{ij} (j = 1,2, \ldots, m)$ into an overall conclusion $\bar{\mathcal{Y}}_i$ applying the $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ operator as below:

$$\bar{\mathcal{Y}}_i = IHF\mathcal{A}_{\mathcal{A}}HA_\delta(H_1, H_2, \dots, H_n) = \oplus_{s=1}^{n}\left(\delta_i H_{\sigma(i)}\right) = \langle 1 - e^{-\left(\sum_{i=1}^{n}\delta_i\left(-\log\left(1-\left(\rho_{\sigma(i)}\right)\right)\right)^{\beth}\right)^{1/\beth}}, e^{-\left(\sum_{i=1}^{n}\delta_i\left(-\log\left(\phi_{\sigma(i)}\right)\right)^{\beth}\right)^{1/\beth}}\rangle.$$

Step 2. Aggregate the score function $Sc(\bar{\mathcal{Y}}_i)$, varied on the total IHF information $(\bar{\mathcal{Y}}_i), (i = 1,2,\dots,n)$ that one can order for the alternative A_i to choose excellent selection A_i. In the event that there is relationship between scores functions $Sc(\bar{\mathcal{Y}}_i)$, then we continue to calculate the accuracy amounts of $Hac(\bar{\mathcal{Y}}_i)$, and on the basis of accuracy, alternative results are ranked.

Step 3. Grade the whole participants A_i on the way to take the best one based on score values on the other way using accuracy value.

The flowchart of the stated algorithm is given in Figure 1.

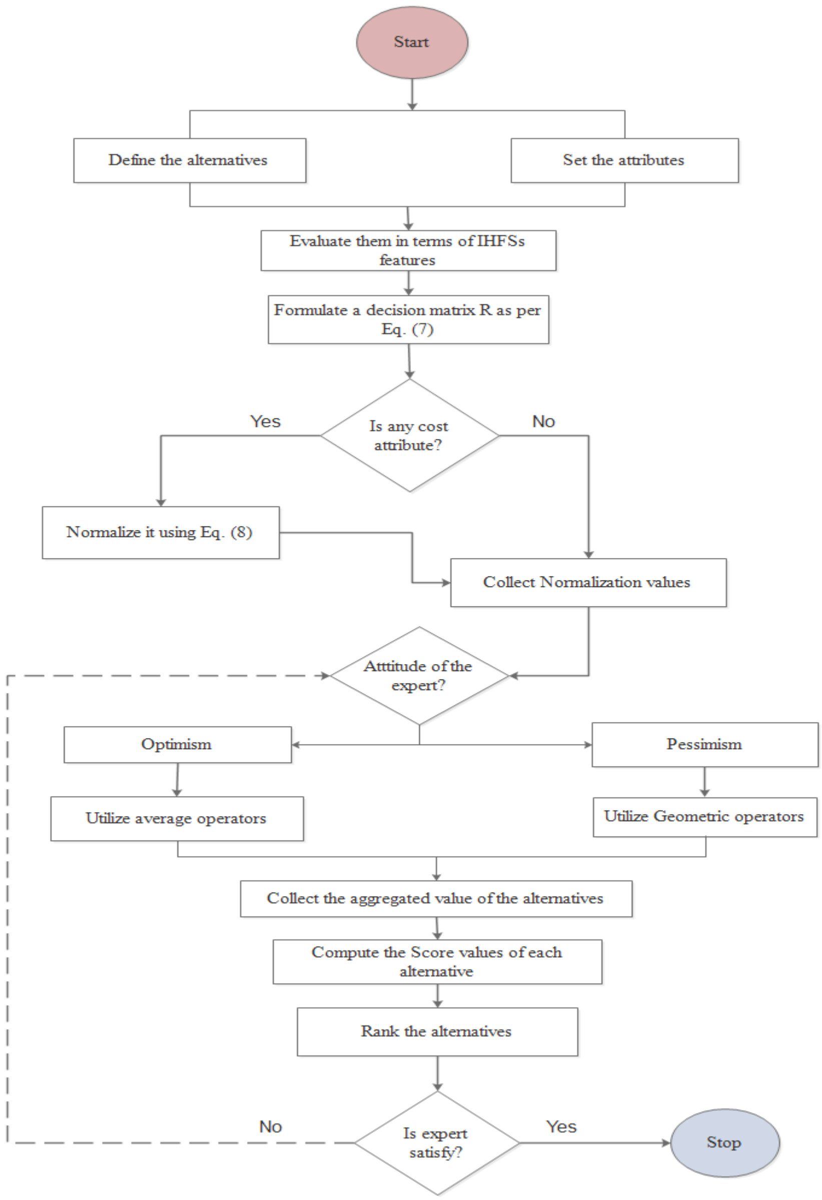

Figure 1. Flow chart of the proposed algorithm.

5. Numerical example

In this section, an investigative example is presented to illustrate the utilization of the proposed strategy in selecting the best mobile phone available on the market. The aim of the proposed approach is to simplify the decision-making process for consumers by providing a systematic and structured method of evaluating options based on their individual needs and preferences.

5.1. Explanation of the problem

In today's world, the mobile phone has become the most essential device for every individual. Various companies offer a wide range of options with varying qualities, making it difficult to determine the best device suitable for an individual's needs. The decision-making process can be daunting and unpleasant for those looking to make a purchase. However, to overcome this challenge, a proposed approach can be utilized to assist the common man in society with making purchasing decisions, whether it be for a mobile phone, car, bungalow, or other products. Let us consider Mr. Noor Zeb, who plans to buy a versatile version of an android. He visits the market, and after pre-screening, he received five different advanced gadgets for advanced assessment. He has to plan based on the four subsequent parameters: (i) G_1 is a long-lasting battery and crystal-clear display. (ii) G_2 is the accessible mobile phone price and attractive in weight and size. (iii) G_3 is the wrap-speed processing and storage capacity. (iv) G_4 is versatile camera and built-in security. The assigned weight is allocated by experts as $\omega = (0.4,0.2,0.1,0.3)^T$ the five devices $A_i(i = 1, \dots 5)$ are to be evaluated in indistinctness with IHF information (chosen from [11]). Table 2 shows the attributes and alternatives details.

Table 2. IHF information table.

Altenative	$\boldsymbol{G_1}$	$\boldsymbol{G_2}$	$\boldsymbol{G_3}$	$\boldsymbol{G_4}$
$\boldsymbol{A_1}$	{{0.1,0.3},{0.1,0.4	{{0.0,0.3},{0.3,0.4}}	{{0.0,0.3},{0.1,0.1}}	{{0.2,0.4},{0.1,0.2}}
$\boldsymbol{A_2}$	{{0.1,0.0},{0.2,0.2	{{0.0,0.1},{0.1,0.2}}	{{0.1,0.1},{0.1,0.3}}	{{0.1,0.2},{0.1,0.3}}
$\boldsymbol{A_3}$	{{0.3,0.2},{0.2,0.1	{{0.1,0.2},{0.0,0.1}}	{{0.2,0.5},{0.2,0.1}}	{{0.0,0.6},{0.2,0.1}}
$\boldsymbol{A_4}$	{{0.3,0.1},{0.2,0.5	{{0.3,0.5},{0.1,0.1}}	{{0.1,0.0},{0.1,0.2}}	{{0.3,0.2},{0.2,0.2}}
$\boldsymbol{A_5}$	{{0.2,0.1},{0.5,0.1	{{0.1,0.4},{0.1,0.2}}	{{0.2,0.2},{0.5,0.2}}	{{0.3,0.5},{0.2,0.2}}

Step 1. Consider that $\beth = 1$, using the IHF$\mathcal{A}_{\mathcal{A}}$WA operator to compute the general alternative values $\bar{\mathcal{Y}}_i(i = 1, \dots, 5)$ of five participants A_i,

$$\mathcal{Y}_1 = (0.100943, 0.507031),$$
$$\mathcal{Y}_2 = (0.038361, 0.48483),$$
$$\mathcal{Y}_3 = (0.121264, 0.398785),$$
$$\mathcal{Y}_4 = (0.115598, 0.509434),$$
$$\mathcal{Y}_5 = (0.121435, 0.533009).$$

Step 2. Aggregate the score numbers $Scr(\mathcal{Y}_i)$ of the general IHFNs of $\mathcal{Y}_i$,

$$Scr(\mathcal{Y}_1) = -0.40609,$$
$$Scr(\mathcal{Y}_2) = -0.44647,$$
$$Scr(\mathcal{Y}_3) = -0.27752,$$
$$Scr(\mathcal{Y}_4) = -0.39384$$
$$Scr(\mathcal{Y}_5) = -0.41157.$$

Step 3. Classify all the five gadgets $A_i(i = 1, \dots 5)$ respectively the result of the score function of the general $Scr(\mathcal{Y}_i)(i = 1,2, \dots, 5)$ IHFNs as

$$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2.$$

From this ranking order, we obtain A_3, which is chosen as the most suitable mobile phone for Mr. Noor Zeb.

5.2. The effect of the parameter $\beth$ in this method

We apply various values of the parameter $\beth$ within the aforementioned methodologies to categorize the five alternatives (A_i) to show the effects of the varied amounts of the parameter $\beth$. In Table 3 and graphically in Figure 2, the ordering implications of the five participants (A_i) using the $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ operator are shown. It is reflected that as the amount of $\beth$ for the $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ operator is improved, then scores of the alternatives also rises regularly. However, the corresponding ordering stays the same (i.e., $A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$). This indicates that the suggested procedures contain the characteristic of isotonicity, allowing the DM to select the best result in accordance with their favorites.

Table 3. Score values obtained by changing parameter.

$\beth$	$Score\,\mathcal{Y}_1$	$Score\,\mathcal{Y}_2$	$Score\,\mathcal{Y}_3$	$Score\,\mathcal{Y}_4$	$Score\,\mathcal{Y}$	Ranking
1	-0.40609	-0.44647	-0.27752	-0.39384	-0.41157	$A_3 \succ A_4 \succ A_1 \succ A_5 \succ A_2$
2	-0.21117	-0.23049	-0.34755	-0.21114	-0.17601	$A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$
3	-0.88225	-0.87951	-0.74294	-0.87521	-0.9093	$A_3 \succ A_4 \succ A_2 \succ A_1 \succ A_5$
4	-0.92782	-0.93167	-0.79014	-0.91952	-0.95151	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
5	-0.95132	-0.95898	-0.81069	-0.94251	-0.97166	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
10	-0.98552	-0.9949	-0.74868	-0.97795	-0.99622	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
20	-0.9947	-0.99976	-0.14491	-0.98994	-0.99978	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$
50	-0.998	-1	0	-0.99601	-1	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_2$

Additionally, as seen in Figure 2, we can deduce that the level of products of the choices are identical whether results of $\beth$ are altered in the example, and the reliable grading outcomes shows the solidity of the suggested $IHF\mathcal{A}_{\mathcal{A}}HA_\delta$ operators.

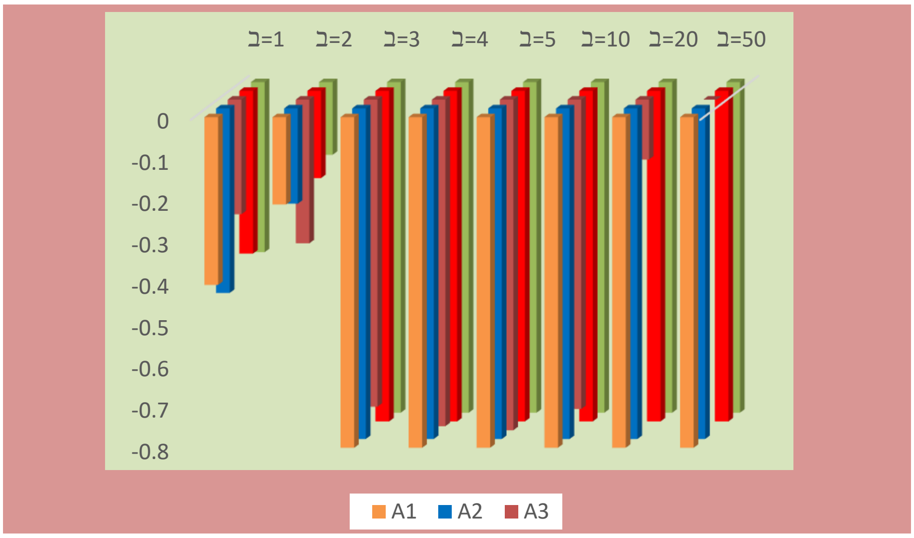

Figure 2. Score values of the alternatives for various values $\beth$ by IHF$\mathcal{A}_{\mathcal{A}}$WA operator.

5.2. Comparative analysis

In the current part, we compared the presented techniques with the $IHFHP$ [11] and $IHFPG$ aggregation operators and $IF\mathcal{A}_{\mathcal{A}}\,H$ [16]. The comparative values are recorded in Table 4 and the results are geometrically represented in Figure 3.

Table 4. Comparative results.

$\boldsymbol{Techniques}$	$\boldsymbol{Score}\,\mathcal{Y}_1$	$\boldsymbol{Score}\,\mathcal{Y}_2$	$\boldsymbol{Score}\,\mathcal{Y}_3$	$\boldsymbol{Score}\,\mathcal{Y}_4$	$\boldsymbol{Score}\,\mathcal{Y}_5$	Ranking
Mahmood et al. [11] IHFPWA	0.2112	-0.0498	0.3706	0.118	0.156	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
Mahmood et al. [11] IHFPWG	0.1897	-0.1476	0.196	0.1008	0.1196	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
Senapati et al. [16] IF$\boldsymbol{\mathcal{A}}_{\mathcal{A}}$WA	-0.17694	-0.38131	-0.13638	-0.22612	-0.21063	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
Proposed Model	-0.92782	-0.93167	-0.79014	-0.91952	-0.95151	$A_3 \succ A_4 \succ A_1 \succ A_2 \succ A_5$

In the following, we employ the established methodology to suggest a prospective evaluation of four potential emerging technology firms for commercialization.

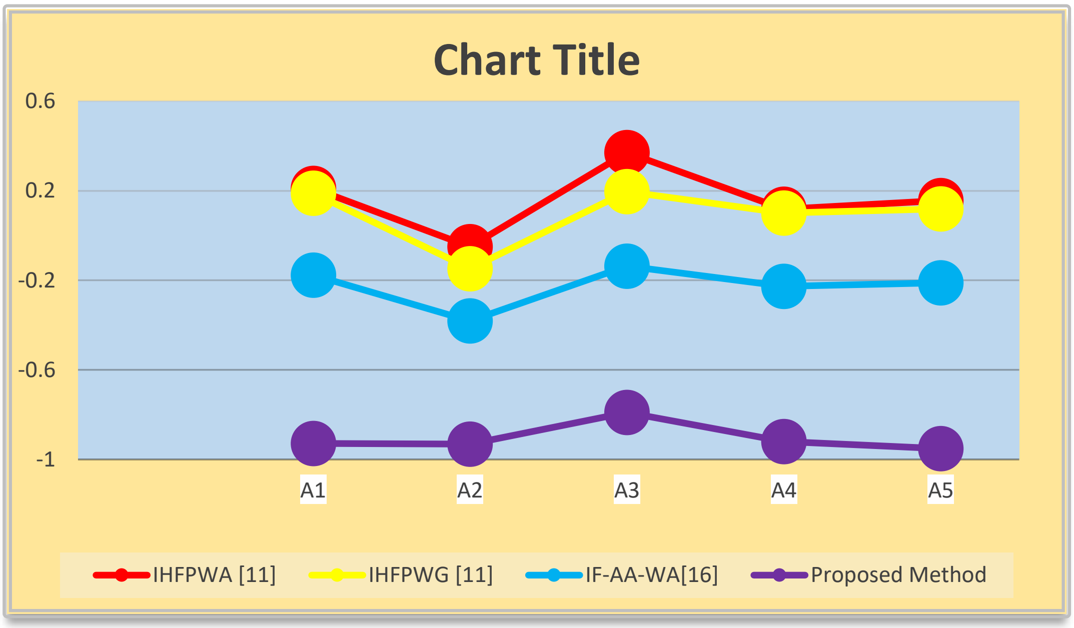

Figure 3. Graphically representation of comparison study.

1) Table 4 and Figure 3 guides that $IFHA$ operators are a specific form of the established $IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}$ model and occurs when the set of MG and set of NMG is taken as a singleton.

Therefore, our developed method is more general in contrast with IFWA operators, as described by Senapati et al. [16].

2) Table 4 and Figure 3 directed that $IHFHA$ operators are a specific form of the established $IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}$ model and comes about by supposing $\beth = 1$. Therefore, our approach is more effective in contrast with IHFWA operators, as described by Mahmood et al. [11].
3) The calculating complexity of our approaches is lesser than existing models such as the IHFWA and $IHFHG$ operators [11]. When this happens, the recommended solutions contain a parameter that may modify the calculated value based on the real decision demands and confines frequent already-existing IHF aggregation operators. Appropriately, the assistance is that the developed model proves a superior intensity of agreement and flexibility.
4) The principal benefit of our suggested model over Mahmood's $IHFHA$ operator is that the $IHF\mathcal{A}_{\mathcal{A}}HA_{\delta}$ operator has the attractive feature of monotonically increasing with respect to the parameter $\beth$, allowing decision-makers to select the proper result with respect to their risk favorites. If the decision-maker prefers risk, we may set the parameter's value as low as is practicably possible; if the decision-maker is risk averse, we can set the parameter's value as high as is practicably possible. As a result, the decision-maker can use the best result of the parameter by their risk tolerance and concrete requirements. The practice described in this assessment is superior to the other existing approaches, according to the judgments and investigation realized above.

6. Conclusions

The article begins by examining the Aczel-Alsina $T.\mathcal{N}$ and $T.C\mathcal{N}$ in the IHF environment and proposes new operating rules for IHFNs while exploring their characteristics. Based on these functional laws, the article introduces exclusive aggregation operators, where the $IHFA_A\, HA_{\delta}$ operator, $IHFA_A\, OHA_{\delta}$ operator, and $IHFA_A\, HA_{\delta}$ operator were designed to conform to the conditions where the allocated opinions are IHFNs. Furthermore, the article investigates the MADM problem using the Aczel-Alsina aggregating operators and IHF data, resulting in various approaches to solve IHF MADM issues. To demonstrate the proposed method's feasibility and effectiveness, the article presents a useful example. Additionally, the article analyzes the parameter of Aczel-Alsina and discusses its effects on the alternatives, and the pictorial form helps to understand the importance of this factor. A comparative analysis is also discussed with existing and proposed approaches, highlighting the established model's benefits in detail, including its geometrical approach.

Although the proposed method provides a broader model to address the decision-making process, which is accompanied by the uncertainty aspect through considering the satisfaction and dissatisfaction degrees of the information, the utilization of the proposed approach on high-dimensional problems still deserves further exploration. To resolve this problem, we intend to create a more adaptable mathematical frameworks in the future, which should allow us to record a noticeable greater range of evaluation. Additionally, we can generalize our approach, which will allow the expanse of the approach of the application to deal with practical cases. Finally, we can range the method to the different types of decision-making difficulties and some new generalizations of fuzzy situations such that Interval-valued IHF data and q-rung orthopair fuzzy sets along with the diverse application on the different areas, such as the multiobjective intelligent model [38], feature extraction [39], etc.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The authors are grateful to King Saud University, Riyadh, Saudi Arabia for funding this work through Researchers Supporting Project Number (RSP2023R18).

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Mardani, A. Jusoh, K. Nor, Z. Khalifah, N. Zakwan, A. Valipour, Multiple criteria decision-making techniques and their applications–a review of the literature from 2000 to 2014, *Econ. Res.-Ekon. Istraživanja*, **28** (2015), 516–571. https://doi.org/10.1080/1331677X.2015.1075139
2. W. Ali, T. Shaheen, H. G. Toor, F. Akram, M. Z. Uddin, M. M. Hassan, Selection of investment policy using a novel three-way group decision model under intuitionistic hesitant fuzzy sets, *Appl. Sci.*, **13** (2023), 4416. https://doi.org/10.3390/app13074416
3. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
4. M. K. Saraji, A. Mardani, M. Köppen, A. R. Mishra, P. Rani, An extended hesitant fuzzy set using SWARA-MULTIMOORA approach to adapt online education for the control of the pandemic spread of COVID-19 in higher education institutions, *Artif. Intell. Rev.*, **55** (2022), 181–206. https://doi.org/10.1007/s10462-021-10029-9
5. S. Qu, Y. Li, Y. Ji, The mixed integer robust maximum expert consensus models for large-scale GDM under uncertainty circumstances, *Appl. Soft Comput.*, **107** (2021), 107369. https://doi.org/10.1016/j.asoc.2021.107369
6. S. Sebastian, T. V. Ramakrishnan, Multi-fuzzy sets: An extension of fuzzy sets, *Fuzzy Inform. Eng.*, **3** (2011), 35–43. https://doi.org/10.1007/s12543-011-0064-y
7. K. T. Atanassov, *On intuitionistic fuzzy sets theory*, Springer, **283** (2012). https://doi.org/10.1007/978-3-642-29127-2
8. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. https://doi.org/10.1002/int.20418
9. H. Garg, A. Keikha, Various aggregation operators of the generalized hesitant fuzzy numbers based on Archimedean t-norm and t-conorm functions, *Soft Comput.*, **26** (2022), 13263–13276. https://doi.org/10.1007/s00500-022-07516-8
10. W. Ali, T. Shaheen, I. U. Haq, H. G. Toor, F. Akram, S. Jafari, et al., Multiple-attribute decision making based on intuitionistic hesitant fuzzy connection set environment, *Symmetry*, **15** (2023), 778. https://doi.org/10.3390/sym15030778
11. T. Mahmood, W. Ali, Z. Ali, R. Chinram, Power aggregation operators and similarity measures based on improved intuitionistic hesitant fuzzy sets and their applications to multiple attribute decision making, *Comput. Model. Eng. Sci.*, **126** (2021), 1165–1187. https://doi.org/10.32604/cmes.2021.014393
12. R. R. Yager, Generalized OWA aggregation operators, *Fuzzy Optim. Decis. Ma.*, **3** (2004), 93–107. https://doi.org/10.1023/B:FODM.0000013074.68765.97

13. R. R. Yager, Prioritized aggregation operators, *Int. J. Approx. Reason.*, **48** (2008), 263–274. https://doi.org/10.1016/j.ijar.2007.08.009
14. H. Zhang, G. Wei, X. Chen, Spherical fuzzy Dombi power Heronian mean aggregation operators for multiple attribute group decision-making, *Comput. Appl. Math.*, **41** (2022), 98. https://doi.org/10.1007/s40314-022-01785-7
15. Z. Xu, R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. Gen. Syst.*, **35** (2006), 417–433. https://doi.org/10.1080/03081070600574353
16. T. Senapati, G. Chen, R. R. Yager, Aczel-Alsina aggregation operators and their application to intuitionistic fuzzy multiple attribute decision making, *Int. J. Intell. Syst.*, **37** (2022), 1529–1551. https://doi.org/10.1002/int.22684
17. N. Ayub, M. Aslam, Dual hesitant fuzzy Bonferroni means and its applications in decision-making, *Ital. J. Pure Appl. Math.*, **48** (2022), 32–53.
18. A. Hadi, W. Khan, A. Khan, A novel approach to MADM problems using Fermatean fuzzy Hamacher aggregation operators, *Int. J. Intell. Syst.*, **36** (2021), 3464–3499. https://doi.org/10.1002/int.22423
19. K. Menger, *Statistical metrics*, In: Proceedings of the National Academy of Sciences of the United States of America, **28** (1942), 535. https://doi.org/10.1073/pnas.28.12.535
20. C. A. Drossos, Generalized t-norm structures, *Fuzzy Set. Syst.*, **104** (1999), 53–59. https://doi.org/10.1016/S0165-0114(98)00258-9
21. G. Deschrijver, C. Chris, E. K. Etienne, On the representation of intuitionistic fuzzy t-norms and t-conorms, *IEEE T. Fuzzy Syst.*, **12** (2004), 45–61. https://doi.org/10.1109/TFUZZ.2003.822678
22. D. Boixader, J. Recasens, Vague and fuzzy t-norms and t-conorms, *Fuzzy Set. Syst.*, **433** (2022), 156–175. https://doi.org/10.1016/j.fss.2021.07.008
23. S. Ashraf, S. Abdullah, M. Aslam, M. Qiyas, M. A. Kutbi, Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms, *J. Intell. Fuzzy Syst.*, **36** (2019), 6089–6102. https://doi.org/10.3233/JIFS-181941
24. M. Pan, J. Li, Q. Yang, Y. Wang, Y. Tang, L. Pan, et al., An adaptive sparse general regression neural network-based force observer for teleoperation system, *Eng. Appl. Artif. Intell.*, **118** (2023), 105689. https://doi.org/10.1016/j.engappai.2022.105689
25. Y. Ji, H. Li, H. Zhang, Risk-averse two-stage stochastic minimum cost consensus models with asymmetric adjustment cost, *Group Decis. Negot.*, 2022, 1–31. https://doi.org/10.1007/s10726-021-09752-z
26. E. P. Klement, R. Mesiar, E. Pap, Generated triangular norms, *Kybernetika*, **36** (2000), 363–377. https://doi.org/10.1007/978-94-015-9540-7
27. E. P. Klement, R. Mesiar, *Logical, algebraic, analytic and probabilistic aspects of triangular norms*, Elsevier, 2005.
28. C. A. Drossos, Generalized t-norm structures, *Fuzzy Set. Syst.*, **104** (1999), 53–59. https://doi.org/10.1016/S0165-0114(98)00258-9
29. J. Aczél, C. Alsina, Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements, *Aequationes Math.*, **25** (1982), 313–315. https://doi.org/10.1007/BF02189626
30. J. Ye, S. Du, R. Yong, Aczel-Alsina weighted aggregation operators of neutrosophic Z-Numbers and their multiple attribute decision-making method, *Int. J. Fuzzy Syst.*, **24** (2022), 2397–2410. https://doi.org/10.1007/s40815-022-01289-w

31. T. Senapati, G. Chen, R. Mesiar, R. R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of Aczel-Alsina triangular norms and their application to multiple attribute decision making, *Exp. Syst. Appl.*, **212** (2023), 118832. https://doi.org/10.1016/j.eswa.2022.118832
32. F. Xu, M. Yan, L. Wang, S. Qu, The robust emergency medical facilities location-allocation models under uncertain environment: A hybrid approach, *Sustainability*, **15** (2023), 624. https://doi.org/10.3390/su15010624
33. M. P. Basílio, V. Pereira, H. G. Costa, M. Santos, A. Ghosh, A systematic review of the applications of multi-criteria decision aid methods (1977–2022), *Electronics*, **11** (2022), 1720. https://doi.org/10.3390/electronics11111720
34. T. Senapati, G. Chen, R. Mesiar, R. R. Yager, Novel Aczel-Alsina operations-based interval-valued intuitionistic fuzzy aggregation operators and their applications in multiple attribute decision-making process, *Int. J. Intell. Syst.*, **37** (2022), 5059–5081. https://doi.org/10.1002/int.22751
35. C. Cheng, Z. Xu, S. Feng, L. Wang, Distribution, ecological risk and source analysis of heavy metals in farmland soil around Chating copper ore in Xuancheng region of southern China, *Pol. J. Environ. Stud.*, **32** (2023), 1581–1594. https://doi.org/10.15244/pjoes/157545
36. L. Wang, Y. Ji, L. L. Zuo, A novel data-driven weighted sentiment analysis with an application for online medical review, *Pol. J. Environ. Stud.*, **31** (2022), 5253–5267. https://doi.org/10.15244/pjoes/151585
37. A. Hussain, K. Ullah, M. S. Yang, D. Pamucar, Aczel-Alsina aggregation operators on T-spherical fuzzy (TSF) information with application to TSF multi-attribute decision making, *IEEE Access*, **10** (2022), 26011–26023. https://doi.org/10.1109/ACCESS.2022.3156764
38. B. Cao, Y. Yan, Y. Wang, X. Liu, J. C. Lin, A. K. Sangaiah, et al., A multiobjective intelligent decision-making method for multistage placement of PMU in power grid enterprises, *IEEE T. Ind. Inform.*, 2022. https://doi.org/10.1109/TII.2022.3215787
39. S. Lu, Y. Ding, M. Liu, Z. Yin, L. Yin, W. Zheng, Multiscale feature extraction and fusion of image and text in VQA, *Int. J. Comput. Intell. Syst.*, **16** (2023), 54. https://doi.org/10.1007/s44196-023-00233-6