Proceedings of CHT-04
ICHMT International Symposium on Advances in Computational Heat Transfer
April 19-24, 2004, Norway
CHT-04-132

SENSITIVITY ANALYSIS FOR THERMAL DESIGN AND MONITORING PROBLEMS OF REFRACTORIES

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ABSTRACT Different methods of sensitivity analysis for thermal design and monitoring problems are described and applied on furnace lining problems. The methods used in this work are based on utilizing *finite differences* and *sensitivity equations*; the former being *code non-invasive* while the latter class of methods is *code invasive.* Due to this fact, finite differences were chosen for the main analysis of the present work. However, on selected modeling parameters, sensitivity equations were used for comparison purposes and showed good agreement with finite differences. The main objective of the work is to determine the location of sensors; e.g., thermocouples, in nonhomogeneous heat conducting materials such that the sensors are as sensitive as possible to thermal changes at positions some distance away from the sensors. Both simplified examples as well as realistic design and monitoring problems of refractories are studied. The results of this study are used in the development of industrial furnace monitoring systems based on the solution of *inverse heat conduction* problems. Due to the nature of these problems, it is of great importance to determine optimal or near-optimal sensor locations, with respect to modeling parameters like the parameters of a critical isotherm and heat transfer coefficients associated with unknown transfer of heat to the surroundings.

NOMENCLATURE

T_a	scaled sensitivity coefficient, $= a\partial T/\partial a$ $[K]$
C	volumetric heat capacity, $= \rho c_p$ $[J/m^3K]$
T^i_∞	bulk temperature of surrounding fluid i $(i = 1, 2)$ $[K]$
T_{crit}	critical temperature (monitoring temperature) $[K]$
a_i	parameter of critical temperature curve $(i = 0, \ldots, 4)$
k_c	thermal conductivity of material c $[W/m\,K]$
h_1/h_2	heat transfer coefficient for cooling mechanism 1/2, $= a_5/a_6$ $[W/m^2\,K]$
Γ_i	boundary curve segment i (i=0,...,4)
R	furnace radius $[m]$
H	height of monitoring region of furnace (vertical intersection) $[m]$
r	space coordinate, radial direction $[m]$
z	space coordinate, height direction $[m]$
b	dependent variable, $= (a_4 - H)/(a_3 - R)$
s	variable curve parameter $(0 \le s \le 1)$
Ω	space region (monitoring intersectional region)

INTRODUCTION

To analyze the behavior of a thermal system, a number of parameters must be specified to characterize the system. These parameters include *material properties* (e.g., thermal conductivity), *geometry* (e.g., shape parameters of a free boundary), and *boundary conditions* (e.g., heat transfer to the surroundings). During the design phase, some of the parameters may change. Often parameters are not known with a high degree of accuracy. A designer may also be free to choose among different materials. Alternative materials and/or conditions might work equally well and could be used interchangeably. There may even be manufacturer-to-manufacturer variability as well as product variability from a single manufacturer. Therefore, we need to play *what if* scenarios with regard to the material properties, geometry and boundary conditions in order to find out about the consequences.

Usually, these scenarios have been set up on an ad hoc basis. Selected parameter values were changed and the analysis repeated. Without a computer, only a limited number of parameter studies could be performed. Computers make it possible to perform a wide range of parameter studies. However, even with a computer these parameter studies still tend to be performed on an ad hoc basis. Based on intuition, the most important parameters would be varied over some range. However, it is possible for even an experienced analyst to miss an important parameter. It is also time consuming to study parameters in an ad hoc manner. Consequently, a more formal procedure needs to be developed. This is where sensitivity analysis comes in.

A goal of thermal design and monitoring system development is to produce robust solutions that function over a wide range of operating conditions. This is particularly true for furnace linings. *Wearline monitoring* of furnace linings should be capable of functioning for both short distance to long distance travelling of "signatures" given to remote sensors (e.g., thermocouples), caused by varying conditions (e.g., lining wear caused by mechanical and/or chemical processes) on the inside of the lining. In addition, the sensors should not be exposed to mechanical loads and high temperatures over long time periods, in order to maximize their functioning life time.

The quality and robustness of a thermal monitoring design can be investigated by means of *sensitivity analysis*, which is introduced and applied in the sections to follow. First, the concepts of sensitivity analysis are given. Then, different methods of performing sensitivity analysis are described. Finally, these methods are applied to the analysis of furnace lining monitoring problems.

SENSITIVITY ANALYSIS

Sensitivity analysis is defined as the study of how variations in input parameters of a computational model cause variations in output. In general, input parameters include geometry, material properties, boundary conditions and initial conditions. Output variables for thermal systems like the ones we are considering here, are temperatures. A measure of this sensitivity is termed the *sensitivity coefficient* and is defined as the partial derivative of the output variable with respect to the parameter of interest. For a general discussion of sensitivity coefficients, see Beck and Arnold [1] and Beck, et al. [2]. Since the focus of this work is on thermal problems, let us define the *scaled first order sensitivity coefficient* of the temperature field with respect to a general parameter a

$$T_a(\mathbf{x}, t; a) \equiv a \frac{\partial T}{\partial a}(\mathbf{x}, t; a) \tag{1}$$

where all parameters other than this parameter are fixed during the differentiation. In Equation (1), $\mathbf{x}$ and t denote space and time coordinates, respectively; and T denotes the temperature.

Figure 1: A $1D$ two-layer slab problem.

Let us also define the *scaled and non-dimensionalized sensitivity coefficient* of the temperature field with respect to a general parameter a

$$\widetilde{T}_a(\mathbf{x},t;a) \equiv \left\{ \frac{\partial T}{\partial a}(\mathbf{x},t;a) - \min_{\mathbf{x},t}[\frac{\partial T}{\partial a}(\mathbf{x},t;a)] \right\} / \left\{ \max_{\mathbf{x},t}[\frac{\partial T}{\partial a}(\mathbf{x},t;a)] - \min_{\mathbf{x},t}[\frac{\partial T}{\partial a}(\mathbf{x},t;a)] \right\} \quad (2)$$

which implies that the values are scaled between 0 and 1. In some cases it is more convenient to use this definition instead of the definition of Equation 1.

The sensitivity coefficient, T_a, is also a field variable in that it depends on position and time just like temperature. In order to understand how one might use the sensitivity coefficient to predict how a system responds if one perturbs the given parameter (e.g., the thermal conductivity), let us expand the temperature field in a Taylor series about a mean value, $\overline{a}$, of the parameter a

$$T(\mathbf{x},t;a) = T(\mathbf{x},t;\overline{a}) + \left.\frac{\partial T}{\partial a}\right|_{\overline{a}} (a-\overline{a}) + \frac{1}{2}\left.\frac{\partial^2 T}{\partial a^2}\right|_{\overline{a}} (a-\overline{a})^2 + \dots \quad (3)$$

One can see that the first order sensitivity coefficient is needed for a first order analysis and that higher order sensitivity coefficients are required for higher order analysis. The present work will focus on first order sensitivity analysis as the computational load scales approximately linearly with the number of parameters. For a second order analysis, the computational load scales as the square of the number of parameters. This may become prohibitive for problems with hundreds of parameters. If the system response and first order sensitivity coefficients are known for the nominal parameter values, equation (3) with higher-order terms neglected, can be used to compute the response at a neighboring point in parameter space. If higher order sensitivity coefficients are required, an initial approach might be to compute second order sensitivity coefficients only for those selected parameters that have large first order sensitivity coefficients.

Sometimes, only the sign of the sensitivity coefficient is important. If for example, the length of a system is increased, does the critical temperature go up or down? In many instances, the sensitivity coefficient is often required as an intermediate step in the solution of parameter estimation, function estimation, uncertainty propagation, and optimization problems. The emphasis of this work is on calculating sensitivity coefficients because they have importance themselves as opposed to just numbers that are fed into a parameter estimation or optimization process.

METHODS OF SENSITIVITY ANALYSIS

Different methods exist for computing the sensitivity coefficients. Here we present three methods; based on *differentiation of analytical solutions*, *finite differences* and the *sensitivity equation*. Other and related methods exist as well (see [3, 6]). In [7] a review of different methods for sensitivity and uncertainty analysis is given.

Differentiation of analytical solutions This method involves differentiating an analytical solution with respect to the parameters of interest. Obviously, this approach is limited to

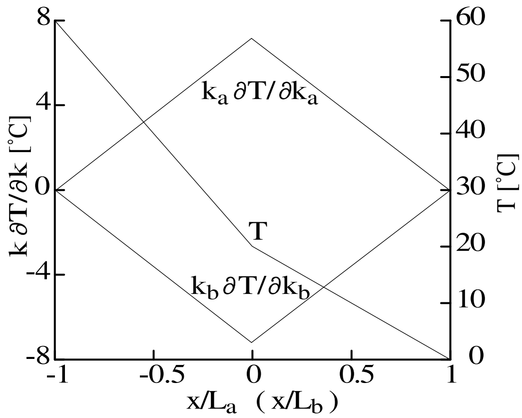

Figure 2: Profiles of temperature and conductivity sensitivity coefficients for the two-layer slab problem.

problems in which analytical solutions are available. However, the method may serve as a tool to verify approximate numerical methods. In order to demonstrate the method consider a $1D$ configuration that might be used in determination of thermal conductivity. The geometry is shown in Figure 1. On the two ends temperatures are kept fixed (T_a and T_b), while the steady state temperature profile is measured. The conductivity of one of the layers is known and the other is to be determined. The analytical expression for the temperature profile is

$$T(x) = \begin{cases} \frac{k_a L_b T_a + k_b L_a T_b}{k_a L_b + k_b L_a} - \frac{k_b L_a (T_a - T_b)}{k_a L_b + k_b L_a} \frac{x}{L_a}, & -L_a \leq x \leq 0 \\ \frac{k_a L_b T_a + k_b L_a T_b}{k_a L_b + k_b L_a} - \frac{k_a L_b (T_a - T_b)}{k_a L_b + k_b L_a} \frac{x}{L_b}, & 0 \leq x \leq L_b \end{cases} \tag{4}$$

The goal in the design of this experiment is to have a large sensitivity to the unknown thermal conductivity and a small sensitivity to the known thermal conductivity. For this simple setup the two thermal conductivity sensitivity coefficients can be computed analytically and are given by

$$T_{k_a} \equiv k_a \frac{\partial T}{\partial k_a} = \frac{k_a L_b k_b L_a}{(k_a L_b + k_b L_a)^2} (T_a - T_b) \cdot \begin{cases} (1 + \frac{x}{L_a}), & -L_a \leq x \leq 0 \\ (1 - \frac{x}{L_b}), & 0 \leq x \leq L_b \end{cases} \tag{5}$$

$$T_{k_b} \equiv k_b \frac{\partial T}{\partial k_b} = -k_a \frac{\partial T}{\partial k_a} \equiv -T_{k_a} \tag{6}$$

The profiles of the temperature and the conductivity sensitivity coefficient are shown in Figure 2. The profiles consist of two straight line segments. Observe that the sensitivity coefficient T_{k_a} is positive throughout, which implies that increasing k_a increases the temperature. The sensitivity coefficient T_{k_b} is negative throughout, which implies that increasing k_b decreases the temperature. The location of maximum thermal conductivity sensitivity T_{k_a} is at the interface between the two layers. Note that the two conductivity sensitivity coefficients are correlated. This means that we can not estimate both thermal conductivities from the same experiment.

The above sensitivity coefficients can be used to choose both the layer thicknesses as well as sensor locations.

The finite difference method In general, analytical solutions to thermal problems of the kind we are studying are not available. However, numerical solution techniques and software are developed to solve realistic thermal problems. The techniques include finite difference, finite volume and finite element methods. We can compute sensitivity coefficients by running the software for two different values of a parameter and use a first-order forward difference or a second-order central difference. Sensitivity coefficients, T_{a_i}, $i = 1, \ldots, n$, are then determined from

$$T_{a_i} = a_i \frac{T(a_1, \ldots, a_i + \Delta a_i, \ldots, a_n) - T(a_1, \ldots, a_n)}{\Delta a_i} + O(\Delta a_i), \tag{7}$$

or

$$T_{a_i} = a_i \frac{T(a_1, \ldots, a_i + \Delta a_i, \ldots, a_n) - T(a_1, \ldots, a_i - \Delta a_i, \ldots, a_n)}{2\Delta a_i} + O(\Delta a_i^2), \tag{8}$$

respectively. This approach requires $n + 1$ solutions for the temperature field and will be first order accurate in Δa_i. If a second order accurate central difference is used instead, then $2n$ solutions of the temperature field will be required. An advantage of this approach is that you numerically solve the problem with different inputs. Since no source code modification is required, the software development costs for this method will be minimal. Any relevant software can be used to accomplish this, provided the computational results are available with sufficient precision.

Numerical experiments are recommended to determine an acceptable value for the finite difference step size Δa. If it is too large, the truncation errors will be unacceptable. On the other hand, if it is too small, computational round-off errors may become a problem. The importance of the latter issue is studied in the following computation of *boundary temperature sensitivity coefficients*. The problem is given by

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) &= 0, \quad r_0 < r < r_1,\ z_0 < z < z_1 && (9)\\
T(r_0, z) &= a, \quad z_0 \le z \le z_1 && (10)\\
T(r_1, z) &= 0, \quad z_0 \le z \le z_1 && (11)\\
\frac{\partial T}{\partial z}(r, z_0) &= 0, \quad r_0 \le r \le r_1 && (12)\\
\frac{\partial T}{\partial z}(r, z_1) &= 0, \quad r_0 \le r \le r_1 && (13)
\end{aligned}$$

This problem was solved numerically using a finite element code. Double precision on a 32-bit machine was utilized. The dimensionless boundary temperature sensitivity coefficients at locations close to the outer wall were computed, utilizing a range of values for $\Delta a_i = a_i - \overline{a}$, $i = 1, \ldots, n$, for a forward difference approximation

$$T_a \equiv a\frac{\partial T}{\partial a} \approx a\frac{T(a_i) - T(\overline{a})}{\Delta a_i} \tag{14}$$

where $\overline{a}$ denotes the reference value. The results are presented in Figure 3.

Slab-wise uniform grids of totally 15×15, 30×30 and 60×60 elements were used. Focusing on the right-hand side of Figure 3, as the relative finite difference step size $(\Delta a/a)$ is decreased down to almost 10^{-6}, the sensitivity coefficients are practically kept constant in each grid case. A further decrease results in deviations for all the three cases. Therefore, roundoff-errors are not a matter of concern in this simple (practically) $1D$ case. More complicated cases, geometrically and/or property-wise, may easily behave in a different manner, resulting in a step-size dependency for much larger values of the step-size.

The sensitivity equation method The sensitivity coefficient is a field variable just like temperature and have its own governing equation. In some cases it is easy to formulate, in other cases it is not. In this section we demonstrate how to derive the field equations for specific

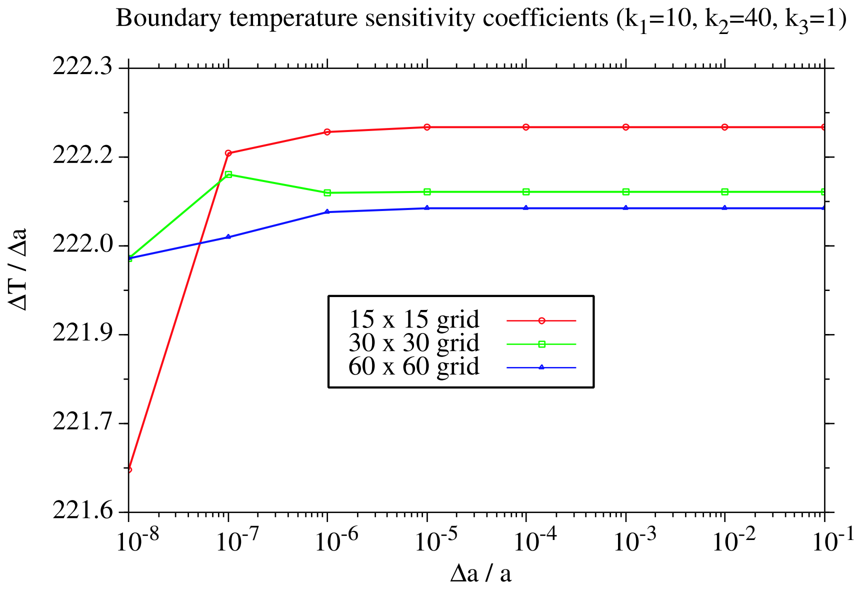

Figure 3: Boundary temperature sensitivity close to the outer wall as a function of $\Delta a/a$ (a is boundary temperature) for three different grid resolutions.

sensitivity coefficients. The formulation involves the differentiation of the governing equation, along with associated initial and/or boundary conditions, with respect to the parameters of interest. These sensitivity equations are then solved numerically, using the same algorithm (and software) used to solve the heat equation. To demonstrate the formulation process, consider a $1D$ planar slab with a temperature boundary condition on one face and heat transfer boundary condition on the other side. This problem is shown schematically in Figure 4. The heat equation and boundary conditions can be written as follows:

$$C\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = 0\,, \quad 0 < x < L\,, \quad 0 < t \leq T \tag{15}$$

$$T = T_b\,, \quad x = 0\,, \quad 0 < t \leq T \tag{16}$$

$$-k\frac{\partial T}{\partial x} = h(T - T_\infty)\,, \quad x = L\,, \quad 0 < t \leq T \tag{17}$$

$$T = T_0\,, \quad 0 \leq x \leq L\,, \quad t = 0. \tag{18}$$

The parameters of this problem are given by the vector

$$p^T = [\,C\ k\ T_b\ h\ T_\infty\ T_0\,]\,. \tag{19}$$

We differentiate Equations (15)-(18) with respect to each element in this parameter vector.

Volumetric heat capacity sensitivity coefficient of the example. Starting with the volumetric

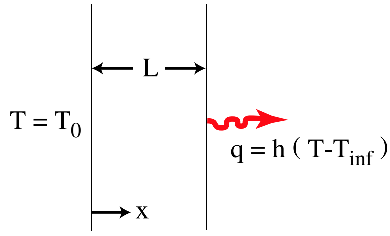

Figure 4: Schematic of $1D$ problem with temperature and heat transfer boundary conditions.

heat capacity sensitivity coefficient, differentiating Equation (15) with respect to C gives

$$\frac{\partial T}{\partial t} + C\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial C}\right) - \frac{\partial}{\partial x}\left\{k\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial C}\right)\right\} = 0 \tag{20}$$

where it has been assumed that the order of differentiation can be interchanged. We introduce the scaled sensitivity coefficient by multiplying Equation (20) through by C to obtain

$$C\frac{\partial T}{\partial t} + C\frac{\partial}{\partial t}\left(C\frac{\partial T}{\partial C}\right) - \frac{\partial}{\partial x}\left\{k\frac{\partial}{\partial x}\left(C\frac{\partial T}{\partial C}\right)\right\} = 0 \tag{21}$$

which is valid if C is independent of T. Note that for *phase change problems*, utilizing *the enthalpy method*, this is not the case. The volumetric heat capacity sensitivity coefficient is easily identified in Equation (21), and its governing equation can be written as

$$C\frac{\partial T_C}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T_C}{\partial x}\right) = -C\frac{\partial T}{\partial t}\,. \tag{22}$$

Equation (22) is the partial differential equation that describes the field variable T_C. Note that the left hand side of this equation is identical in form to that of the original heat equation. However, there is a source term on the right hand side that was not present in the heat equation. If the temperature field is known, then this source term is a known function of position and time.

We continue the development of the equations governing the behavior of T_C by differentiating the boundary and initial conditions with respect to C. Differentiating Equations (16)-(18) with respect to C, we have

$$T_C = 0\,, \quad x = 0, \quad 0 < t \le T \tag{23}$$

$$-k\frac{\partial T_C}{\partial x} = hT_C\,, \quad x = L, \quad 0 < t \le T \tag{24}$$

$$T_C = 0\,, \quad 0 \le x \le L, \quad t = 0. \tag{25}$$

This assumes again that the temperature field is known prior to the computation of the sensitivity field and that C is independent of x.

The formulation of the field equation and associated boundary and initial conditions for T_C is complete and is given by Equations (22)-(25). Due to the similarities in form of the heat equation and the volumetric heat capacity sensitivity equation, the same technique can be used to numerically solve these equations. It does not matter if the discretization algorithm is finite difference, finite volume or finite element. In fact, the existing software coding used to include the effects of capacitance, diffusion and source terms for the heat equation can be used to form the analogous terms for the sensitivity equation. The computational procedure is first time marching the heat equation one time step and then to solve the sensitivity equation. The source term for T_C in Equation (22) is known from the temperature solution.

From Equation (22), *the time rate of change of temperature drives the volumetric heat capacity sensitivity field*. If the temperature field is not changing with time, the volumetric heat capacity sensitivity will tend toward zero. For a problem with a positive temperature rise rate, this source term is negative, suggesting a negative sensitivity to the volumetric heat capacity. Similarly, for a body that is cooling, the source term is positive which suggests a positive sensitivity to the volumetric heat capacity. As one can see, insight into the thermal response can be gained by simply studying the describing equation for the sensitivity coefficients. In some cases, trends may be determined without actually solving the sensitivity equations. However, it is best to continue the process and numerically solve the sensitivity equations in order to gain maximum insight.

Thermal conductivity sensitivity coefficient of the example. The next component of the parameter vector, the thermal conductivity, is now considered. Following the same procedure as described above, the governing equation for T_k can be written as

$$C\frac{\partial T_k}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T_k}{\partial x}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) . \tag{26}$$

Again, the left hand side of the T_k equation is identical in form to the heat equation; the right hand side has a fictitious source term that is equal to the negative of the gradient of the heat flux; i.e., *gradients in local heat flux drive the thermal conductivity sensitivity field.*

Differentiating the boundary and initial conditions with respect to k, we get the following conditions

$$T_k = 0\,, \qquad x = 0\,, \quad 0 < t \le T \tag{27}$$

$$-k\frac{\partial T_k}{\partial x} = hT_k + k\frac{\partial T}{\partial x}\,, \qquad x = L\,, \quad 0 < t \le T \tag{28}$$

$$T_k = 0\,, \quad 0 \le x \le L\,, \quad t = 0. \tag{29}$$

Among the components of the parameter vector in Equation (19), the volumetric heat capacity C and the thermal conductivity k are special in that they both appear in the governing differential equation for the temperature. For their respective sensitivity equations, inhomogeneous terms are present. For all other parameters that do not appear in the heat equation, their sensitivity governing equation can be written as

$$C\frac{\partial T_a}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T_a}{\partial x}\right) = 0, \qquad a \ne C, k \tag{30}$$

where T_a is the sensitivity coefficient for parameter a; note that this is a homogeneous equation.

The boundary conditions for the remaining parameters of the example. The boundary conditions for the four remaining parameters in Equation (19) will now be addressed. Differentiating the $x = 0$ boundary condition given by Equation (16) with respect to these parameters results in

$$T_a = \left\{ \begin{array}{ll} T_b, & a = T_b \\ 0, & \text{otherwise } (a = h, T_\infty) \end{array} \right\}, \quad x = 0\,, \quad 0 < t \le T \tag{31}$$

Differentiating the $x = L$ boundary condition given by Equation (17) with respect to these parameters results in

$$-k\frac{\partial T_a}{\partial x} = \left\{ \begin{array}{l} hT_a, \quad a = T_b \\ h(T_a + T - T_\infty), \quad a = h \\ h(T_a - T_\infty), \quad a = T_\infty \end{array} \right\}, \quad x = L\,, \quad 0 < t \le T \tag{32}$$

We have given the initial conditions for both T_C and T_k, by Equation (25) and Equation (29), respectively (they are both zero). It is easy to see that if the parameter of interest is anything other than the initial temperature itself, the initial condition for T_a will be zero. The initial conditions for the remaining parameters can be summarized as follows:

$$T_a = \left\{ \begin{array}{ll} a, & a = T_0 \\ 0, & \text{otherwise } (a = T_b, h, T_\infty) \end{array} \right\}, \quad 0 \le x \le L\,, \quad t = 0 \tag{33}$$

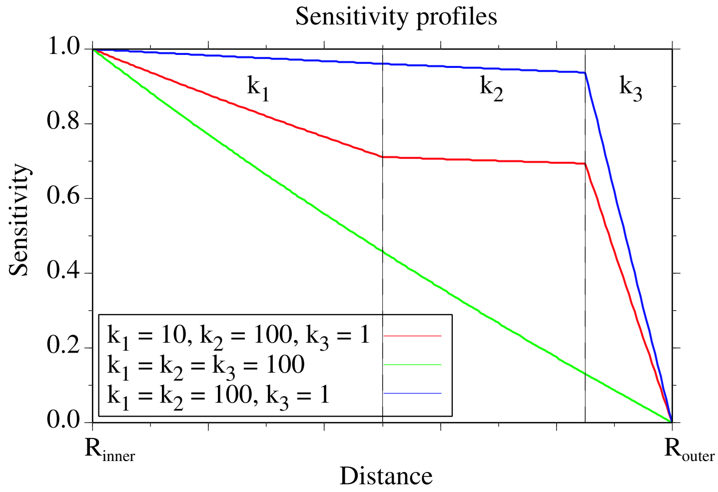

Figure 5: Example of how sensitivity varies with material composition.

Sensitivity coefficients related to temperature (T_b, T_∞, T_0), may need to be scaled by a temperature change instead of the temperature itself. This eliminates the problems with zero temperature when absolute temperature units are not used. These reference temperatures represent a characteristic temperature change for the problem (e.g., the maximum temperature rise of the system, $T_{\max} - T_0$).

ANALYSIS OF FURNACE LININGS

Different methods of thermal sensitivity analysis have been described. The methods are utilized to analyze the behavior of a number of industrial furnace lining designs. The main objective of the analysis is to determine the location of sensors in a nonhomogeneous heat conducting material such that the sensors are as sensitive as possible to thermal changes at positions some distance away from the sensors. Both simplified examples as well as realistic design and monitoring problems are considered in the next sections. This work is part of the overall efforts of improving a thermal design and monitoring system for industrial furnace linings, based on the principle of inverse heat conduction [8,9].

Composite materials We now considered the computation of a *boundary temperature sensitivity coefficient* for the problem of steady $2D$ heat conduction within a composite material. The objective is to find a composition that maximizes the sensitivity of temperature near one part of the boundary, to temperature variations on the opposite side of the domain. The problem is given by Equations (9)-(13). Actually, these equations define a $1D$ problem, and are chosen that way for simplicity reasons. However, the following analysis is not restricted to such situations. Also, for simplicity reasons the composition is restricted to configurations as shown in Figure 5, i.e. a heat conducting domain composed of three different materials placed side-by-side, horizontally.

It is easy to show that the boundary temperature sensitivity coefficient $T_a = a\partial T/\partial a$ satisfies the same set of equations, i.e. Equations (9)-(13) (T replaced by T_a). Solving this sensitivity equation problem numerically for three different compositions, results in the sensitivity profiles ($z = 0.5$) shown in Figure 5.

If temperature sensors are to be placed on the right-hand side of the given composite material, these results indicate that the optimal position of the sensors is at the interface between mate-

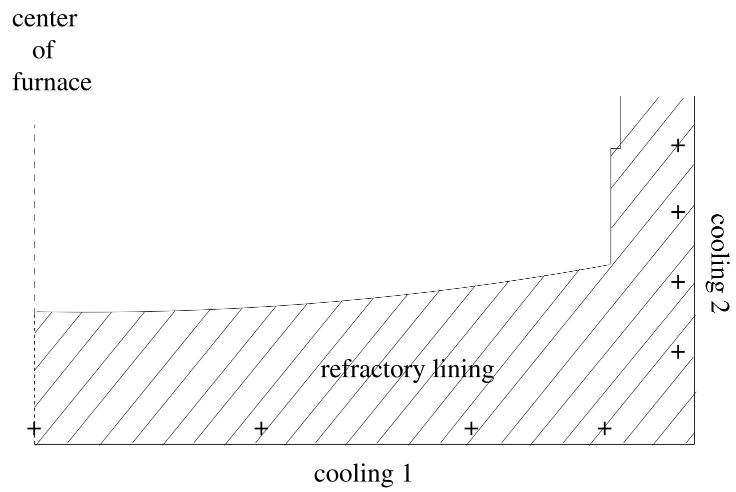

Figure 6: A typical furnace lining with temperature sensors (marked with crosses).

rial 2 and 3 (see Figure 5), or as close as possible to that interface from any practical viewpoint.

Monitoring problems A strategy for monitoring the wear-line of a melting furnace has been developed within an industrial framework. Temperature sensors at different locations in the lining of the furnace, combined with an inverse heat conduction model, are utilized for monitoring the state of the lining. The wear-line boundary is represented by *a critical isotherm*, and the estimation of this isotherm is the crucial part of the problem. For details about this system it is referred to [8,9,10].

For the system to function satisfactory it is of great importance that the heat conducting material is equipped with temperature sensors in an optimal or near optimal way. The sensors have to be located far from the critical isotherm due the high temperatures in these areas as well as other (practical) reasons. In most practical situations the sensors are located close to the outer walls of the linings. A typical situation is shown in Figure 6.

The inverse heat conduction is modelled by utilizing a cubic spline curve to represent the critical isotherm. This is illustrated in Figure 7. The critical isotherm is described by a parametric cubic curve

$$\mathbf{r}(s) = \mathbf{c}_0 + \mathbf{c}_1 s + \mathbf{c}_2 s^2 + \mathbf{c}_3 s^3, \qquad 0 \le s \le 1 \tag{34}$$

where $\mathbf{r}(s) = [r(s), z(s)]^T$. The endpoints A and B (see Figure 7) are obtained with the parameter values 0 and 1, respectively. The coefficients are given by

$$\begin{aligned}
\mathbf{c}_0 &= \mathbf{r}(0) &&= [0, a_2] \\
\mathbf{c}_1 &= \mathbf{r}_s(0) &&= [a_0, 0] \\
\mathbf{c}_2 &= -3[\mathbf{r}(0) - \mathbf{r}(1)] - 2\mathbf{r}_s(0) - \mathbf{r}_s(1) &&= [-2a_0 - ba_1 + 3a_3, -a_1 - 3a_2 + 3H] \\
\mathbf{c}_3 &= 2[\mathbf{r}(0) - \mathbf{r}(1)] + \mathbf{r}_s(0) + \mathbf{r}_s(1) &&= [a_0 + ba_1 - 2a_3, a_1 + 2a_2 - 2H]
\end{aligned} \tag{35}$$

The expression of b is chosen such that the tangent to the spline curve at $s = 1$ is normal to the uppermost bounding line (stippled line). This way the assumption of zero heat flux through this boundary segment is a fairly good approximation. The corresponding direct problem is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0, \quad (r,z)\in\Omega \tag{36}$$

$$-k\frac{\partial T}{\partial r} = 0, \quad \text{on } \Gamma_0 \tag{37}$$

$$-k\frac{\partial T}{\partial z} = h_1(T-T_\infty^1), \quad \text{on } \Gamma_1 \tag{38}$$

$$-k\frac{\partial T}{\partial r} = h_2(T-T_\infty^2), \quad \text{on } \Gamma_2 \tag{39}$$

$$-k\frac{\partial T}{\partial n} = 0, \quad \text{on } \Gamma_3 \tag{40}$$

$$T = T_{crit}, \quad \text{on } \Gamma_4 \tag{41}$$

The Method of Finite Differences is used to compute the sensitivity coefficients $T_{a_i},\ i=0,\ldots,6$. To compare with the Sensitivity Equation Method, the results for the a_5 and a_5 are shown for this method as well.

In the following figures are shown typical temperature distribution sensitivities with respect to the geometry ($T_{a_i},\ i=0,\ldots,4$) and heat transfer coefficient parameters (T_{a_5} and T_{a_6}). These fields are important for the positioning of temperature sensors in the furnace to be monitored. In general, sensors should be located in areas where the sensitivity field has large values, in order to get *a large as possible* signal of the change. In most cases (for the geometry parameters) this implies that the sensors should be located as deep as possible into the linings. To compute the boundary conditions on the outside (bottom and sidewall) it is preferable to have the sensors at the interface between the cooling facility and the lining. In the Figure 8 are also shown *normalized sensitivity field of the sum of all parameter sensitivities* which indicate positions that are preferable for the complete set of model parameters. Note that the direction of the heat flow is important with respect to the sensor locations.

Using the Sensitivity Equation Method on this problem is somewhat more complicated for the 5 curve parameters $a_i,\ i=0,\ldots,4$. For these parameters the sensitivity field problem becomes a *free boundary problem*. Therefore, for $a_i,\ i=0,\ldots,4$, as well as for the a_5 and a_6 parameters, the Finite Difference Method is used. The results are shown in Figures 9 to 13.

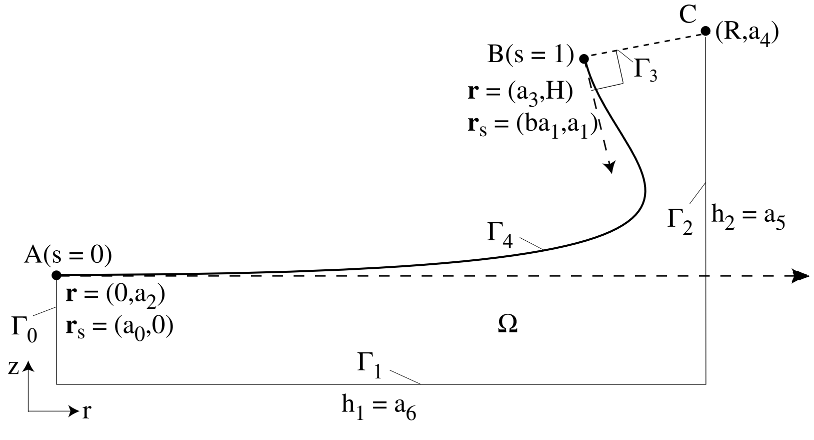

Figure 7: Cubic spline curve representing the critical isotherm to be monitored. The expression of b is $(a_4-H)/(a_3-R)$.

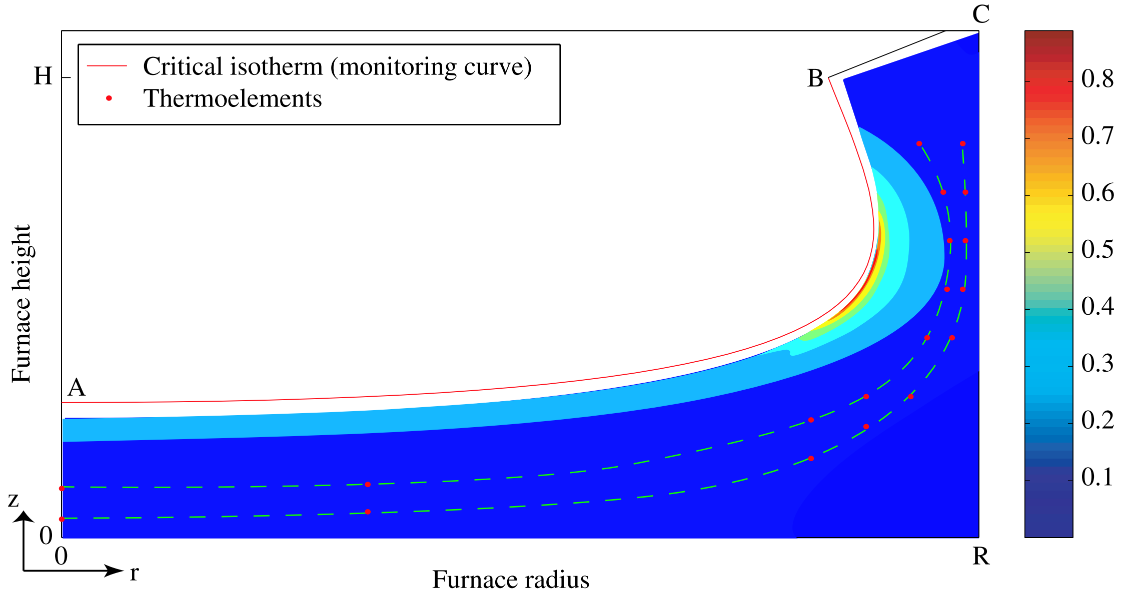

Figure 8: Normalized sensitivity field of the sum of all parameter sensitivities of $a_0 - a_6$, all computed by the Finite Difference Method. The stippled curves are levels of depth for sensors found to be good with respect to sensitivity of model parameters, and given practical requirements for the localization of sensors.

The two parameters $a_5 = h_2$ and $a_6 = h_1$ may be analyzed by the Sensitivity Equation Method. The sensitivity field of a_5 and a_6 are shown in Figures 16 and 17 for comparison purposes. These fields are not scaled, but show qualitatively good agreement with the scaled results of the Finite Difference Method as shown in Figures 14 and 15.

SUMMARY

Three methods for computing sensitivity coefficients have been described, and example calculations were presented for the case of refractory lining design and monitoring. The methods can be divided into two categories; *code invasive* and *code non-invasive*. The *Finite Difference Method* is non-invasive and probably the most general; it can be applied when the source code is not available. This means it can also be used in conjunction with commercially available software. An objection to the finite differences approach is that for non-linear problems, such as problems with temperature dependent properties, each perturbed parameter solution is a non-linear solve. If the same problem is solved using the *Sensitivity Equation Method*, which is code invasive, the sensitivity coefficient equations are linear equations. For complex non-linear and multi-dimensional problems this often caused large savings in computational time compared to the finite differences approach. No matter which method is chosen to be the primary method, *differentiation of analytical solutions* is an important part of the process of verifying that the equations have been implemented correctly.

The sensitivity fields shown in the Figure 8 and the Figures 9 to 14 are simplified models and sensitivity fields as compared to models and analysis of real industrial furnace problems. However, this type of modeling and analysis have shown to be very valuable, both with respect to the design and monitoring of furnace refractories.

ACKNOWLEDGMENTS

This work is supported by *Ferrolegeringsindustriens Forskningsforening* (FFF), *Tinfos Titan & Iron* (TTI) and *Eramet Norway*. We would like to acknowledge the special interest and support from Haavard Elstad and Arne Hildal at TTI, and Johan Gulbrandsen at Eramet.

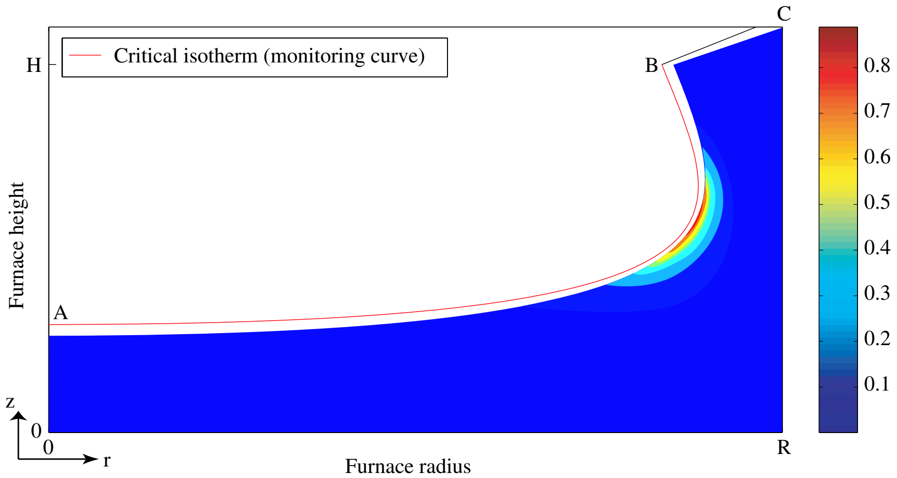

Figure 9: Normalized sensitivity field $\partial T/\partial a_0$ of a_0 (dr/ds at point A) computed by the Finite Difference Method.

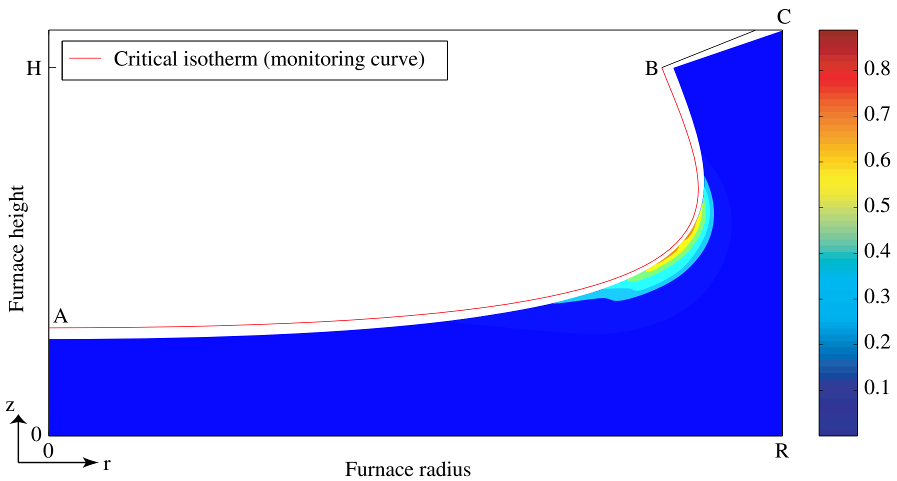

Figure 10: Normalized sensitivity field $\partial T/\partial a_1$ of a_1 (dz/ds at point B) computed by the Finite Difference Method.

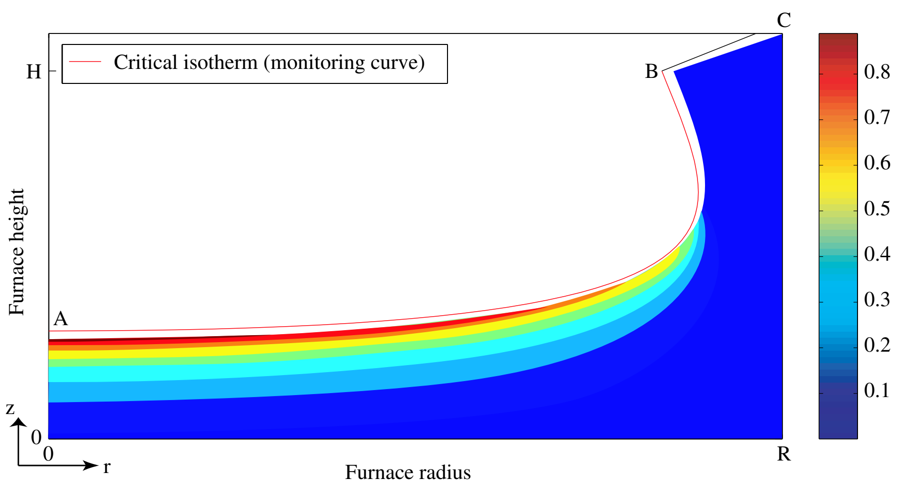

Figure 11: Normalized sensitivity field $\partial T/\partial a_2$ of a_2 (z at point A) computed by the Finite Difference Method.

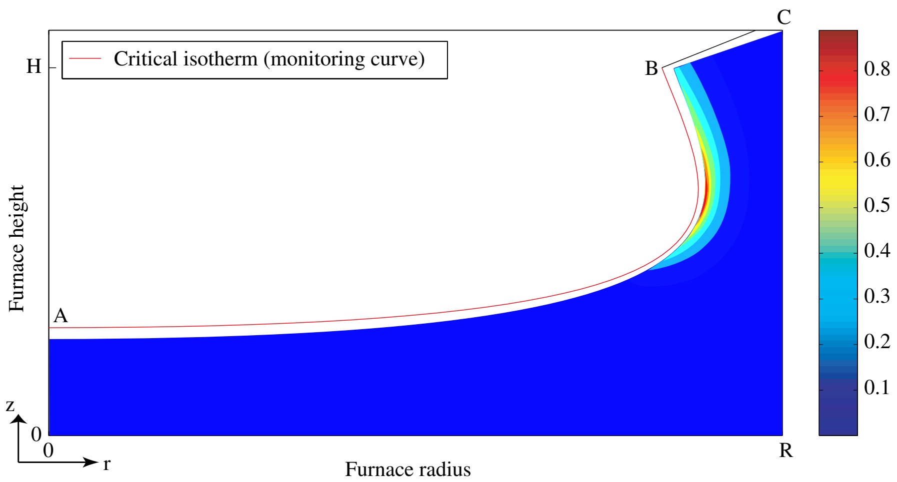

Figure 12: Normalized sensitivity field $\partial T/\partial a_3$ of a_3 (r at point B) computed by the Finite Difference Method.

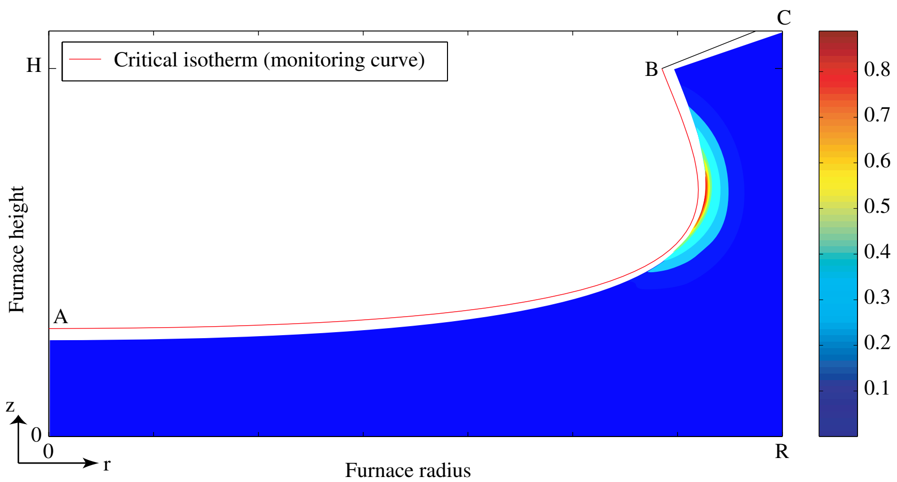

Figure 13: Normalized sensitivity field $\partial T/\partial a_4$ of a_4 (z at point C) computed by the Finite Difference Method.

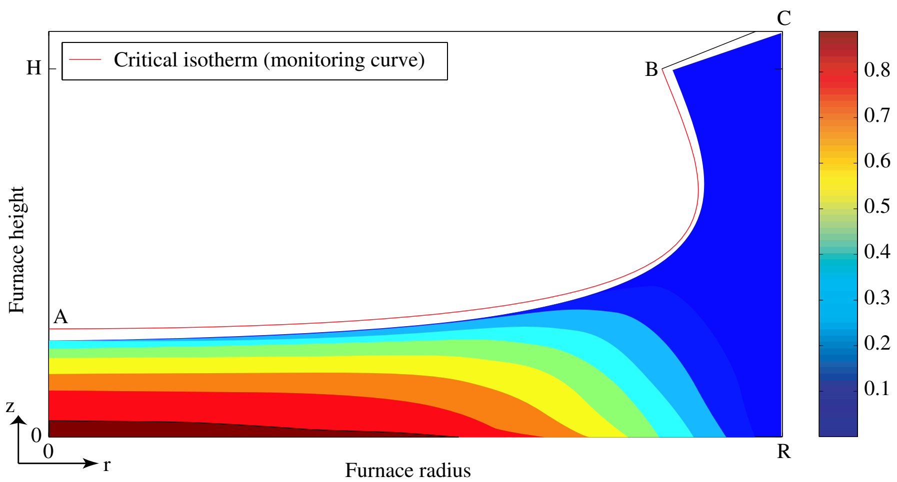

Figure 14: Normalized sensitivity field $\partial T/\partial a_6$ of a_6 (h_a (bottom heat transfer coefficient) along $z = 0$) computed by the Finite Difference Method.

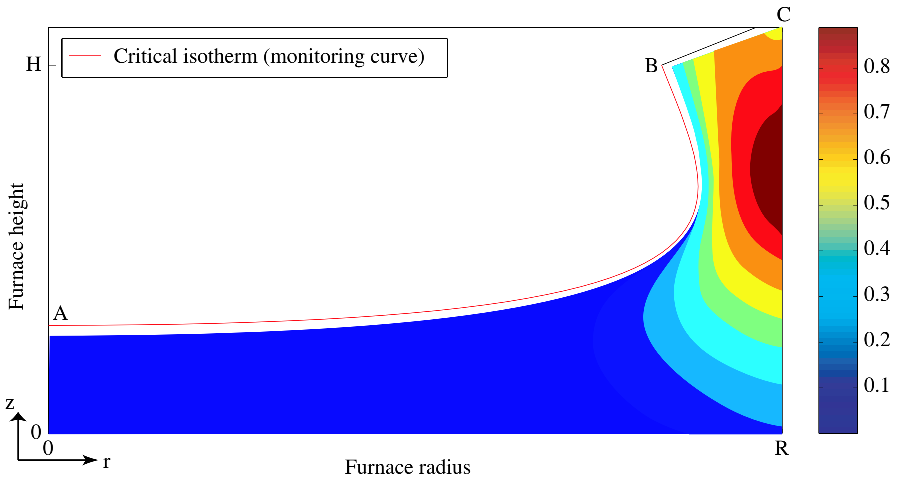

Figure 15: Normalized sensitivity field $\partial T/\partial a_5$ of a_5 (h_w (wall heat transfer coefficient) along $r = 7$) computed by the Finite Difference Method.

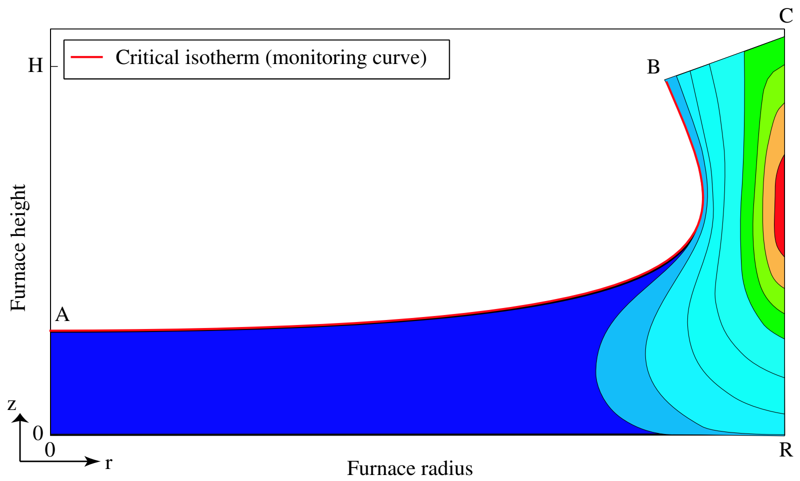

Figure 16: Wall heat transfer coefficient ($h_w = a_5$) sensitivity field computed by the Sensitivity Equation Method. Included for comparison purposes.

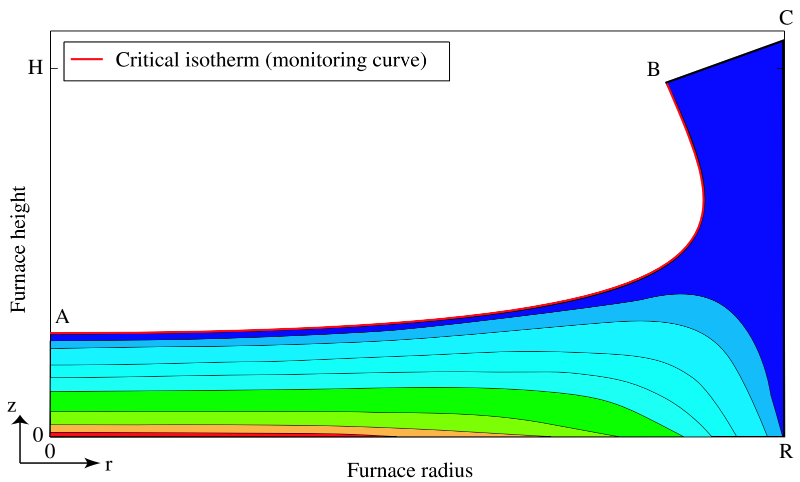

Figure 17: Bottom heat transfer coefficient ($h_a = a_6$) sensitivity field computed by the Sensitivity Equation Method. Included for comparison purposes.

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