

# **Analysis of stress distribution in shear walls by the finite element displacement method**

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Ein umfangreiches Programm für Strukturuntersuchungen eines ebenen Spannungs- und Belastungszustandes wird beschrieben. Das Programm beruht auf der von Argyris, Clough, Turner u. a. angegebenen Methode. Das im Programm verwendete Rechenprinzip ist neu und stellt eine zusammenfassende Verbesserung verschiedener veröffentlichter Rechenprinzipien dar. Einer der hauptsächlichsten Gründe für die Aufstellung des Programmes war die Untersuchung der Spannungsverteilung in schubbelasteten Wänden. Von speziellem Interesse sind die Schubspannungen in vertikalen Verbindungen bei Wänden aus vorfabrizierten Betonelementen in mehrstöckigen Gebäuden. Einige Rechenbeispiele werden angegeben, die den Einfluß von Tür- und Fensteröffnungen auf das strukturelle Verhalten der Wand demonstrieren.

### Synopsis

A computer program for plane stress or plane strain analysis of structures is described. The program is based on the finite element displacement method. As computational element is used a new type of triangular element, with 9 nodal points and 2 internal displacement modes. Numerical examples illustrate the accuracy of the method, and its application in analysis of shear walls.

### Introduction

A number of methods exist for analysis of shear walls /1,2,3/. Generally they are based on one-dimensional elastic theory, with the usual assumptions on stress and strain distribution made in this theory. Regular vertical rows of door and window openings are taken into account by making further simplifying assumptions. Among the advantages of these methods is that they can be used in hand computations. Among the disadvantages are the uncertainties in the basic assumptions, and that irregular systems (i. e. scattered door and window openings) cannot be properly analysed.

Shear walls are essentially plane structures in a state of plane stress. In recent years, the finite element method has to a great extent been applied successfully to analysis of such structures. However, in the author's opinion, the previously known element types would not be a satisfactory basis for a computer program specifically designed to analyse shear walls. Element types with completely compatible displacement fields would result in inadmissible overestimation of the stiffness of certain parts of the structure (i. e. door beams), or would require such a large number of computational elements that the capacity of available computers would be exceeded /8/, /10/. The convergence characteristics of element types based on displacement continuity in nodal points only were considered uncertain /6,9,12/. Therefore, the author has derived the stiffness matrix of a new type of triangular element.

### Derivation of Stiffness Matrix

The basic idea of the element displacement method is to regard the structure as an assembly of a finite number of elements, interconnected in a finite number of nodal points. The stiffness of the structure is represented by its nodal point stiffness matrix  $K$ , and is obtained by adding the individual element stiffness matrices in an appropriate manner.  $K$  serves as coefficient matrix in a system of simultaneous equations. The equations express the principle of minimum potential energy for the structure, which is equivalent to the equilibrium conditions in all the nodal points. The solution vector consists of all nodal point displacements.

To find the element stiffness matrix, one assumes for the element a finite number of displacement modes, characterised by nodal point displacements. Because the final displacement pattern of the element will be restricted to a linear combination of the assumed modes, it is important that these include all essential modes. - The strains in the element are found as partial derivatives of displacements. Application of the appropriate stress - strain relationship (Hooke's law), and of the principle of minimum po-

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tential energy, then yields the stiffness matrix. Fig. 1 shows a triangular element with 9 nodal points. The displacement modes

$$u(x, y) = f_j(x, y) \text{ and } v(x, y) = f_j(x, y) \quad j = 1, 2, 3 \quad (1)$$

are characterised by the relative horizontal and vertical displacements of the corner nodal points 1, 2, 3. These are the traditional modes assumed for triangular elements.

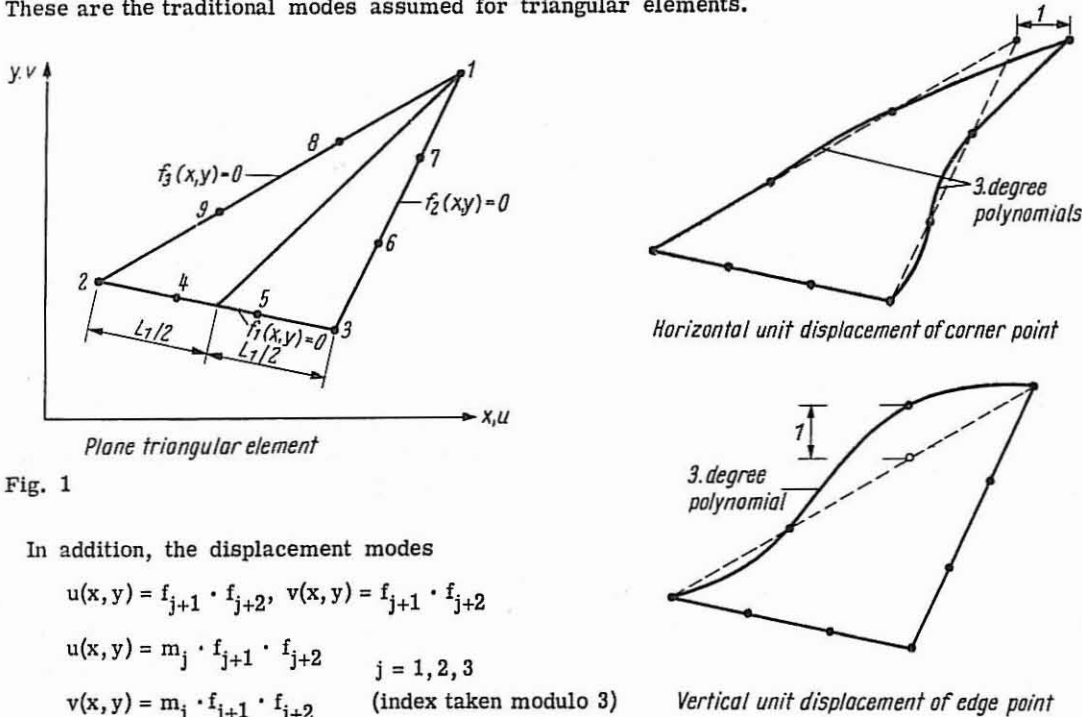


Fig. 1

In addition, the displacement modes

$$u(x, y) = f_{j+1} \cdot f_{j+2}, \quad v(x, y) = f_{j+1} \cdot f_{j+2}$$

$$u(x, y) = m_j \cdot f_{j+1} \cdot f_{j+2} \quad j = 1, 2, 3$$

$$v(x, y) = m_j \cdot f_{j+1} \cdot f_{j+2} \quad (\text{index taken modulo } 3)$$

are allowed for. They are characterised by the displacements of the edge nodal points 4 through 9, relative to the displacements given by the traditional modes.

Finally, the internal modes

$$u(x, y) = f_1 \cdot f_2 \cdot f_3 \text{ and } v(x, y) = f_1 \cdot f_2 \cdot f_3 \quad (3)$$

are assumed. They are kinematically independent from the displacements of the 9 nodal points, but could be characterised by displacements of an internal point.

The assumed displacement functions imply that both displacement components are represented as third-degree polynomials in one variable along element edges. When continuity is imposed in 4 points on the common edge of two adjacent elements, full kinematic compatibility along the whole edge will therefore be ensured. - The assumptions also imply that computed stresses in an element are represented by second-degree polynomials in two variables, whereas the traditional assumptions for triangular elements imply constant stresses within an element, and more recent methods assume linearly varying stresses /7, 10/.

On the basis of the displacement assumptions, the 20 x 20 element stiffness matrix  $k$  is computed in the machine, and the following relation can be set up:

$$k \cdot v = \begin{bmatrix} k_{ii} & k_{ie} \\ k_{ei} & k_{ee} \end{bmatrix} \begin{bmatrix} v_i \\ v_e \end{bmatrix} = S = \begin{bmatrix} S_i \\ S_e \end{bmatrix} \quad (4)$$

The submatrices with indices  $i$  and  $e$  correspond to internal and external nodal point displacements and forces.  $S$  is the vector of forces acting on the element, and  $v$  the nodal point displacement vector. At this stage of the computations,  $S$  only contains forces corresponding to distributed loads in the interior of the element (i. e. own weight).

Before setting up the global stiffness matrix  $k$  and load matrix  $R$ , the internal element displacements  $v_i$  are eliminated. Equation (4) is then reduced to

$$k_{ee \text{ red.}} \cdot v_e = S_{e \text{ red.}} \quad (5)$$

where

$$k_{ee \text{ red.}} = k_{ee} - k_{ei} k_{ii}^{-1} k_{ie} \quad (6)$$

and

$$S_e \text{ red.} = S_e - k_{ei} k_{ii}^{-1} S_i \quad (7)$$

The static significance of inclusion and elimination of internal modes is perhaps best recognised if the procedure is regarded as an application of the principle of minimum potential energy. The stresses computed from arbitrarily assumed displacements are generally not in equilibrium internally in the element. The minimum potential energy principle is equivalent to the equilibrium conditions, and its application on the internal displacement modes results in the best approximation to equilibrium also locally in the interior of the element. - Numerical computations have shown that the reduction leads to very significant decreases in the element stiffnesses. As the reduction leads to very significant decreases in the element stiffnesses. As the resulting stiffness still can be shown to be an upper bound to the theoretically correct stiffness, the inclusion of the internal modes represents an equally significant improvement in the method.

### Computer Program and numerical Results

The method outlined above has been incorporated in a computer program. The program is operating on the Univac 1107 computer, and is mainly written in the FORTRAN IV programming language. Input to the program consists of geometry and topology of the structure, elastic properties of the material, loads and boundary conditions. The maximum program capacity today is 450 elements, which is sufficient for numerous practical applications.

The present program assumes linearly elastic stress - strain relationship, with completely arbitrary anisotropy provided for. The finite element method can, however, also be used with any other assumptions. This would require some additional programming, and considerably more computer time for problem solution.

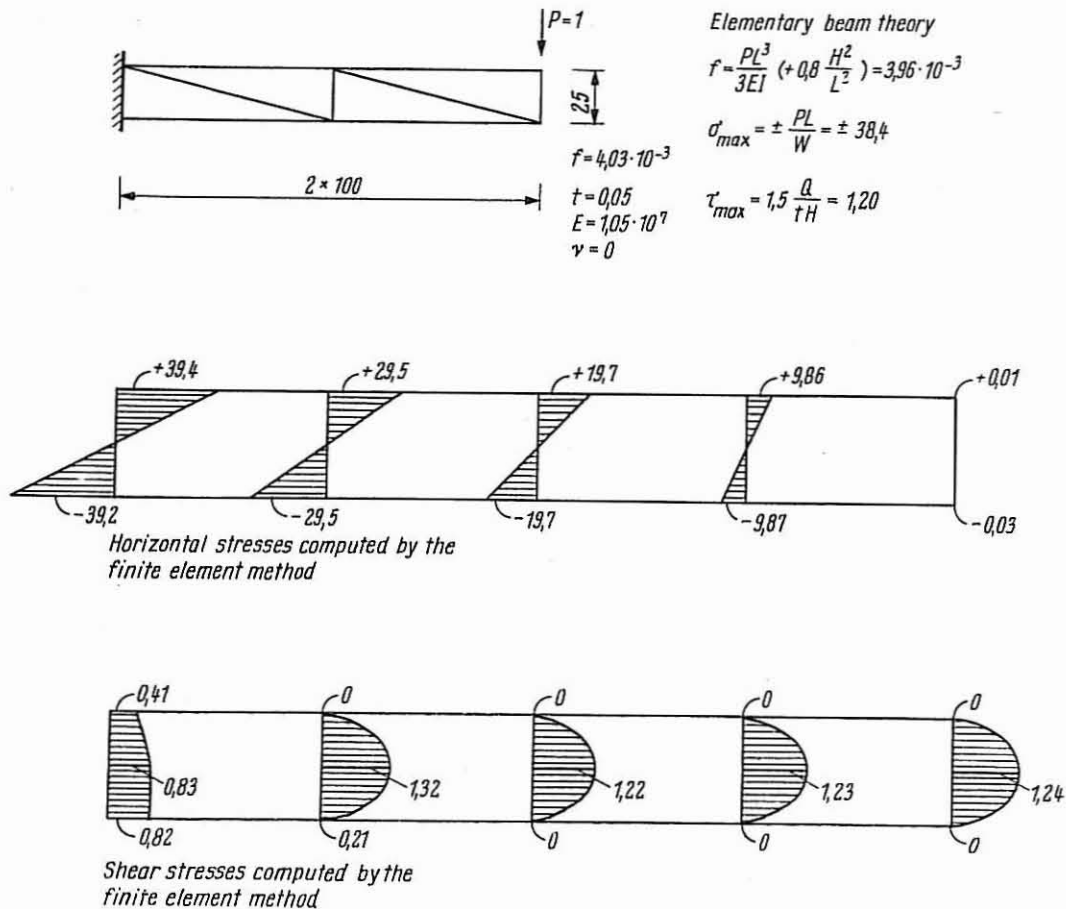


Fig. 2



For each loading case, output from the computer consists of two parts:

1. Displacements of all nodal points in the x and y directions.
2. Stresses in the element. For each element, stresses in the corners and edge midpoints are given. For each point, the  $\sigma_x$ ,  $\sigma_y$  and  $\tau_{xy}$  stress components are printed, as well as the principal stresses and their direction.

Fig. 2 shows a cantilever beam with a height/length ratio of 1/8, and with a constant shear force. This is of course a simple one-dimensional problem where use of more sophisticated methods is normally not justified. It has, however, proved to be a tough test for finite element representations with completely compatible displacements. - The beam is divided into four triangular elements of the new type. Computed stresses and tip deflection are seen to be in good agreement with elementary beam theory.

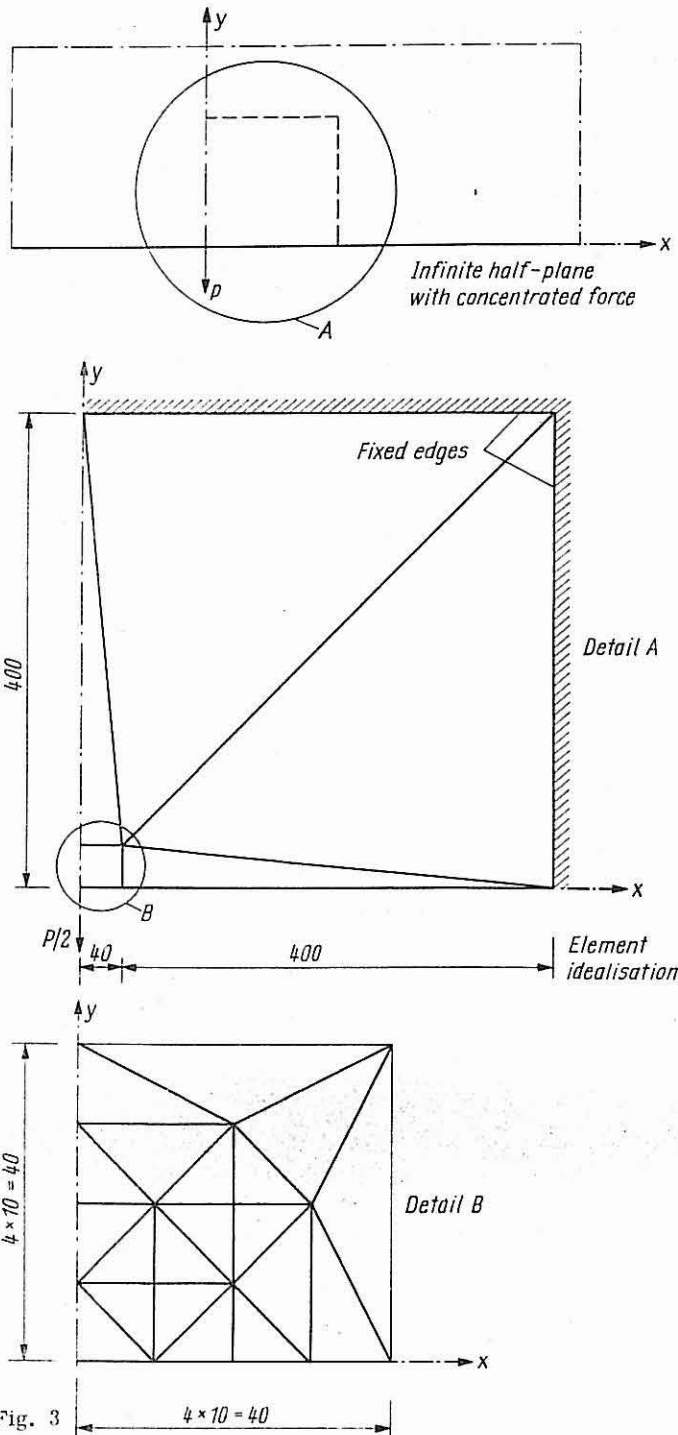


Fig. 3

Table 1:  $\sigma_y / \frac{P}{t} \cdot 10^{-2}$

y \ x	Method	0	5	10	15	20
20	A	3,18	2,82	2,04	1,30	0,80
	F.E.M.	3,18	2,87	2,01	1,30	0,76
15	A	4,24	3,44	2,03	1,06	0,58
	F.E.M.	4,10	3,62	2,01	1,05	0,52
10	A	6,37	4,07	1,59	0,60	0,25
	F.E.M.	7,01	4,14	1,58	0,52	0,23
5	A	12,73	3,18	0,51	0,13	0,04
	F.E.M.	15,08	5,83	0,69	0,08	0,02

Table 2:  $\sigma_x / \frac{P}{t} \cdot 10^{-2}$

y \ x	Method	0	5	10	15	20
20	A	0	0,18	0,51	0,73	0,80
	F.E.M.	-0,12	0,02	0,38	0,61	0,69
15	A	0	0,38	0,90	1,06	0,98
	F.E.M.	-0,16	0,12	0,72	0,89	0,83
10	A	0	1,02	1,59	1,36	1,02
	F.E.M.	-1,09	1,00	1,39	1,15	0,81
5	A	0	3,18	2,04	1,15	0,70
	F.E.M.	0,08	4,05	1,70	1,00	0,41

Table 3:  $\tau_{xy} / \frac{P}{t} \cdot 10^{-2}$

y \ x	Method	0	5	10	15	20
20	A	0	0,70	1,02	0,98	0,80
	F.E.M.	-0,02	0,69	0,98	0,96	0,77
15	A	0	1,15	1,36	1,06	0,73
	F.E.M.	-0,04	1,11	1,25	1,05	0,72
10	A	0	2,04	1,59	0,90	0,51
	F.E.M.	-0,70	3,13	1,48	0,84	0,50
5	A	0	3,18	1,02	0,38	0,18
	F.E.M.	3,54	4,25	0,92	0,38	0,17

Fig. 4 A: Analytic  
F.E.M.: Finite element method  
 $\sigma_y$  in singular point according to  
F.E.M. =  $37,86 \cdot 10^{-2} \frac{P}{t}$

Shear wall with wind load

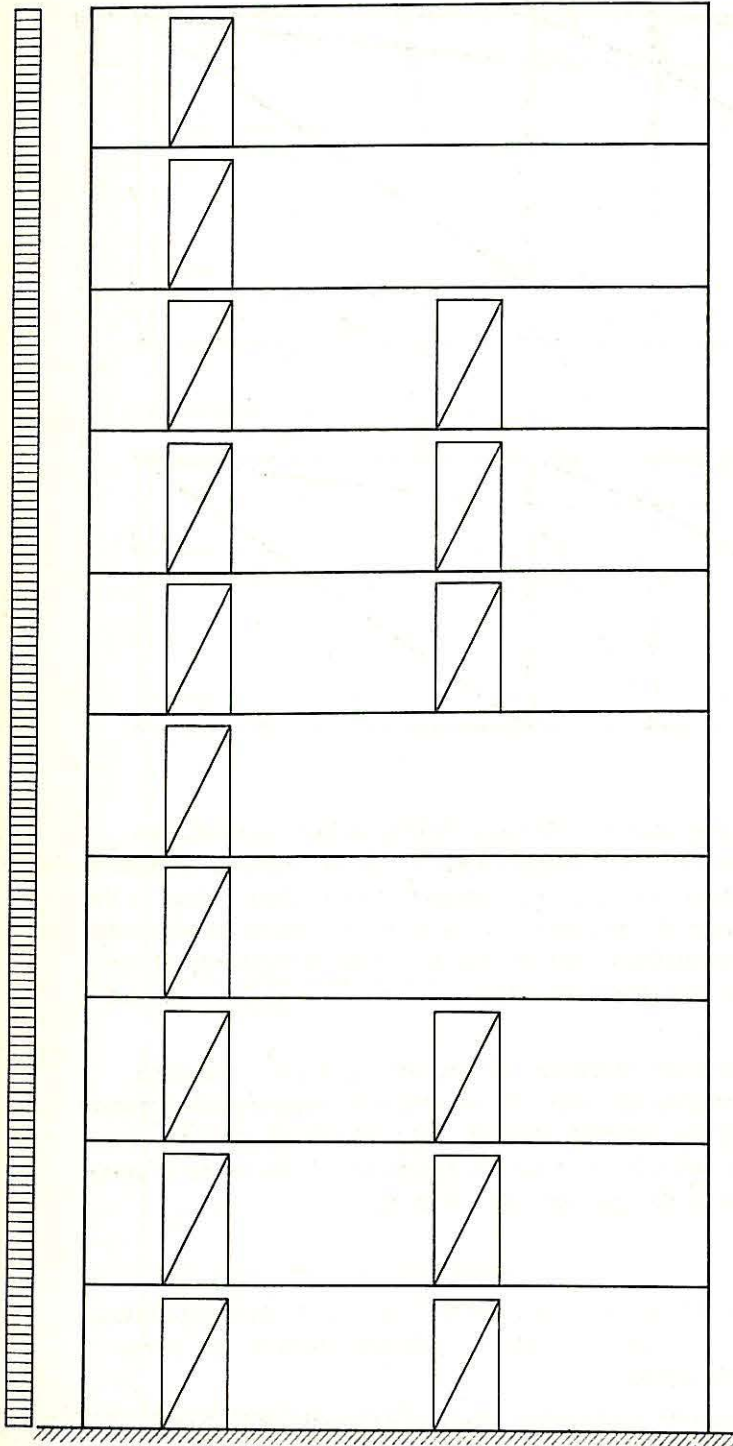


Fig. 5

Vertical stress distribution

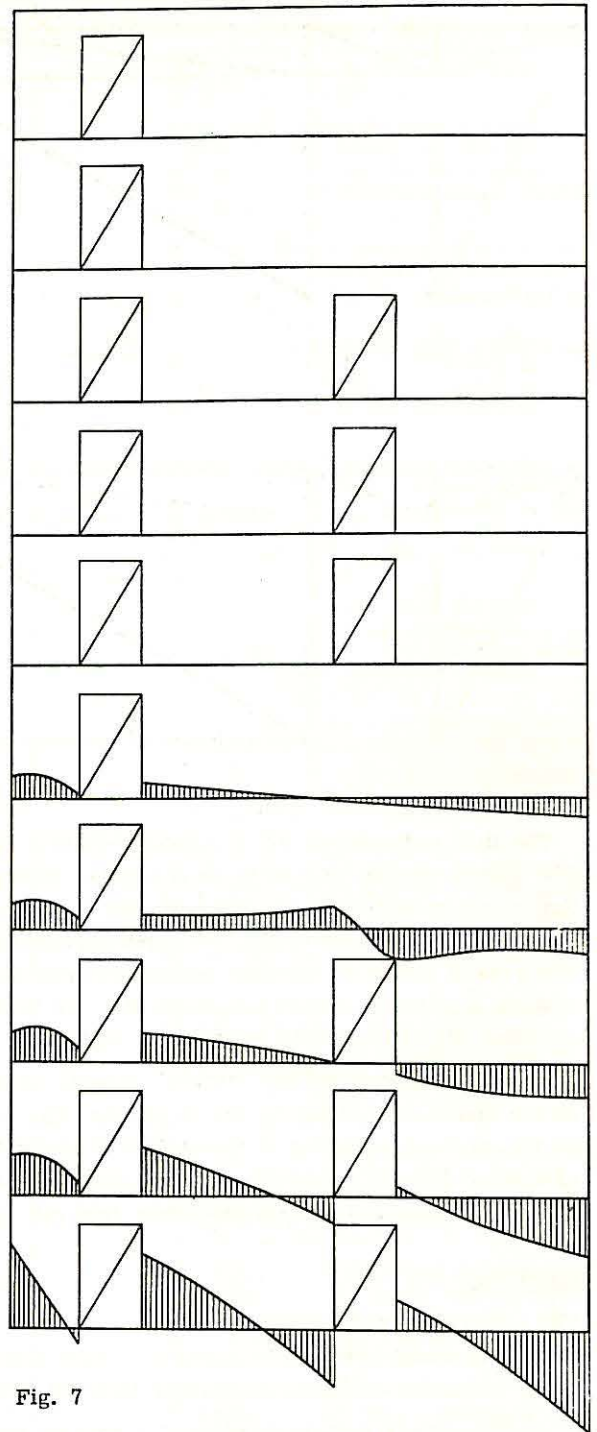


Fig. 7

The second example is a typically two-dimensional problem. Fig. 3 shows an infinite half-plane, with a concentrated force normal to the free edge. The figure also shows how the problem is simulated by dividing a finite part of the plate into triangular elements. The tables in fig. 4 show the stresses computed analytically and with the finite element method in an area close to the singular point. The stresses given for the element method in any point simple averages of the element stresses in the point. In most points no significant difference between the element stresses occur. - Only in the points closest to the singular point differences of any significance between analytic and element method results can be observed.

## ELEMENT IDEALISATION IN STORIES

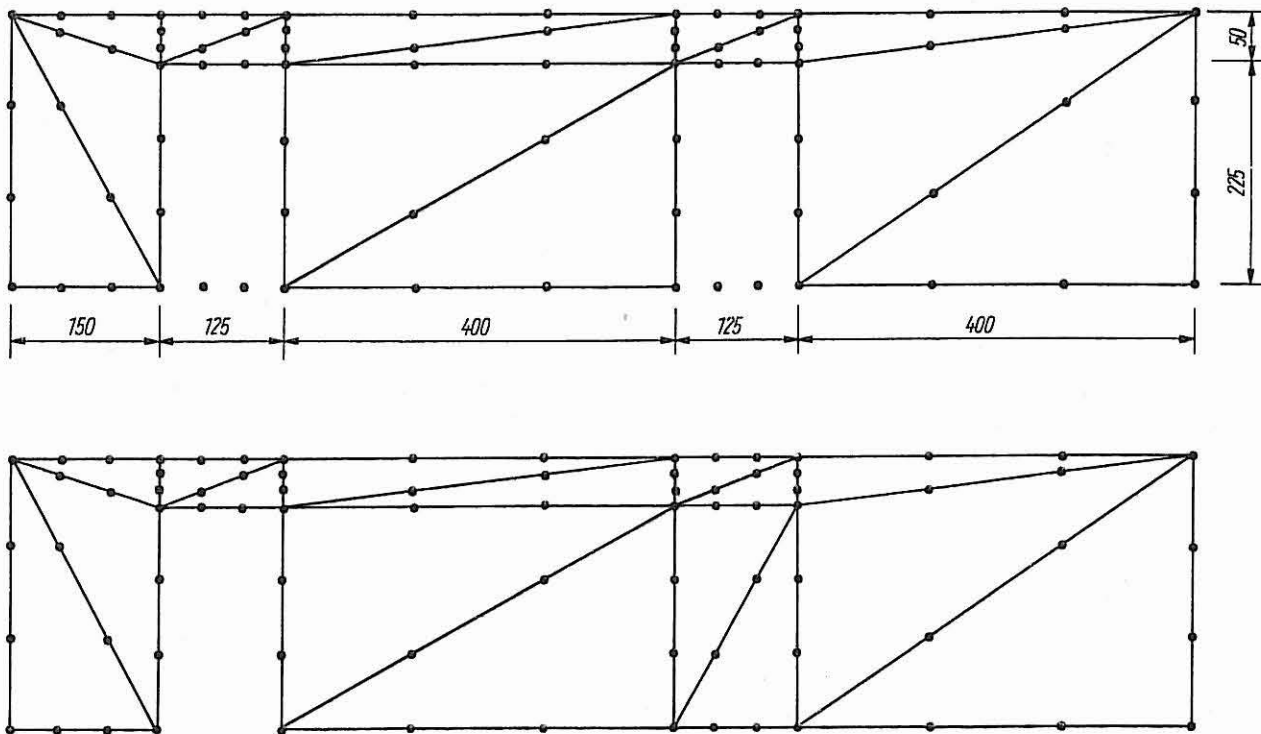


Fig. 6

The final example, shown in figures 5-8, is a shear wall in a 10-story building subject to wind load. The wall is divided into 14 to 16 triangular elements for each story. To solve this problem, the computer had to set up and solve a system of nearly 1300 simultaneous, linear equations. Total solution time on the computer was 13 min. 16 sec., of which approximately  $9\frac{1}{2}$  min. was spent solving the system of equations. The results cannot be checked against accepted computational methods because of the irregularity of the system. A number of equilibrium checks have been carried out by numerical integration of the computed stresses, all of them with satisfactory results.

In fig. 7 diagrams for the vertical stresses at different levels are shown, and fig. 8 gives the shear stress distribution along a vertical section. The irregular openings are seen to have considerable influence on the stress distribution in the wall. - In prefabricated concrete element walls, the shear stresses are of special interest, since the vertical joints have a reduced shear-carrying capacity. If one wants to assign a reduced shear stiffness to the joints, this can easily be built into the program.

### Conclusion

The numerical example serve to verify the method and the program, and to demonstrate their capabilities in solving complicated plane stress problems such as shear wall analysis. The new element type introduced has some additional advantages that are worth noting:

1. The greatly reduced number of necessary elements in structure idealisation significantly facilitates the setup of a computational model and the preparation of input data for the program.
2. With previous element types it has frequently been difficult to interpret the computed stresses. This difficulty is greatly diminished with the new elements.

Both these factors serve to unburden the human program user, and to facilitate the practical application of computers in structural computations. Further improvements in program capacity, efficiency and flexibility of use are now being considered.

Shear stress distribution in vertical section

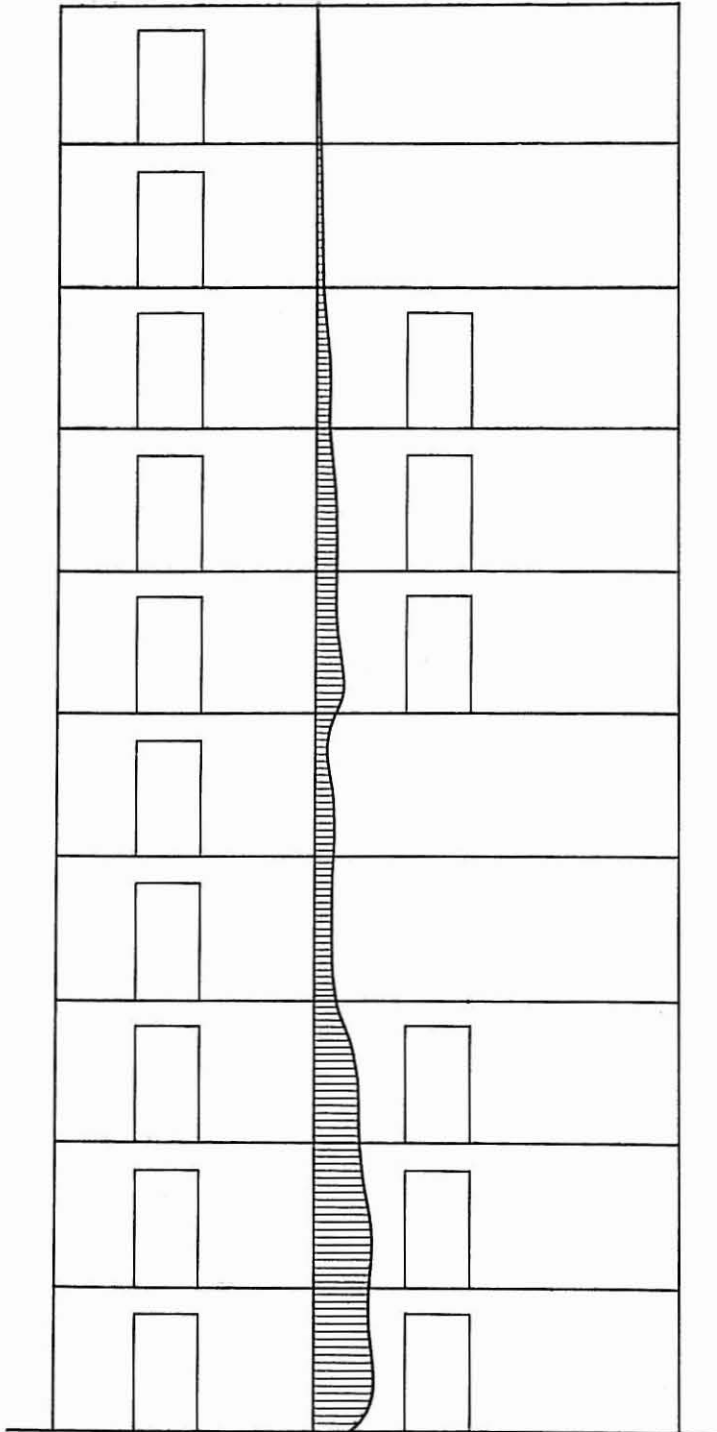


Fig. 8

References :

- /1/ Rosman, R.: Beitrag zur statischen Berechnung waagrecht belasteter Querwände bei Hochbauten. Bauing. 35 (1960) 4, S. 133-136.
- /2/ Baehre, R.: Vinstabiliserande väggar uppbyggda av delement. Danderyd, Sweden 1960. 2 vol. (Arne Johnson Ingenjörbyrå. Tekniska meddelanden, nos. 4-5).
- /3/ Eriksson, Owe: Analysis of wind bracing walls in multi-storey housing. Ingeniören, International edition, vol. 5, no. 4 (1961) p. 115-124.
- /4/ Turner, M. J.; Clough, R. W.; Martin, H. C., and Topp, L. J.: Stiffness and deflection analysis of complex structures. Journal of the aeronautical sciences, vol. 23, no. 9 (1956).
- /5/ Clough, R. W.: The finite element method in plane stress analysis. American Society of Civil Engineers. Conference on Electronic Computation, 2. Pittsburgh, Pa. 1960. Conference papers, p. 345-378.
- /6/ Clough, R. W.: The finite element method in structural mechanics. In: Zienkiewicz, O. C., and Holister, G. S., editors. Stress analysis. London, Wiley, 1965, p. 85-119.
- /7/ Veubeke, B. Fraeijs de: Displacement and equilibrium models in the finite element method. In: Zienkiewicz, O. C., and Holister, G. S., editors. Stress analysis. London, Wiley, 1965, p. 145-197.
- /8/ Argyris, J. H.; Kelsey, S., and Kamel, H.: Matrix methods of structural analysis. In: Veubeke, B. Fraeijs de. ed. Matrix methods of structural analysis. Oxford, Pergamon, 1964, p. 1-164 (AGARDograph, 72)
- /9/ Gallagher, Richard h.: A correlation study of methods of matrix structural analysis. With contributions by Ivan Rattinger and John S. Archer. Oxford, Pergamon, 1964, 113 p. (AGARDograph, 69).
- /10/ Argyris, J. H.: Triangular elements with linearly varying strain for the matrix displacement method. Journal of the Royal Aeronautical Society, vol. 69, no 658 (1965) p. 711-713.
- /11/ Argyris, J. H.: Reinforced fields of triangular with linearly varying strain; effect of initial strains. Journal of the Royal Aeronautical Society, vol. 69, no. 659 (1965) p. 799-801.
- /12/ Melosh, R. J.: Basis for derivation of matrices for the direct stiffness method. Journal of the American Institute of Aeronautics and Astronautics, vol. 1, no. 7 (1963) p. 1631-1637.



