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ON THE DISPERSIVE WAVE-DYNAMICS OF 2×2 RELAXATION SYSTEMS

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Abstract. We consider hyperbolic conservation laws with relaxation terms. By studying the dispersion relation of the solution of general linearized 2×2 hyperbolic relaxation systems, we investigate in detail the transition between the wave-dynamics of the homogeneous relaxation system and that of the local equilibrium approximation.

We establish that the wave velocities of the Fourier components of the solution to the relaxation system will be monotonic functions of a stiffness parameter $\varphi = \varepsilon\xi$, where ε is the relaxation time and ξ is the wave number. This allows us to extend in a natural way the classical concept of the sub-characteristic condition into a more general *transitional* sub-characteristic condition.

We further identify two parameters β and γ that characterize the behavior of such general 2×2 linear relaxation systems. In particular, these parameters define a natural transition point, representing a value of φ where the dynamics will change abruptly from being equilibrium-like to behaving more like the homogeneous relaxation system. Herein, the parameter γ determines the location of the transition point, whereas β measures the degree of smoothness of this transition.

Keywords: relaxation; wave velocities; sub-characteristic condition.

1. Introduction

We are interested in the wave-dynamics of hyperbolic conservation laws with relaxation terms. Such a system consisting of N equations in one spatial dimension can in general be written in the relaxation form

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{1}{\varepsilon}\boldsymbol{Q}(\boldsymbol{U}), \tag{1.1}$$

where $\boldsymbol{U} = \boldsymbol{U}(x,t) \in G \subseteq \mathbb{R}^N$ for some state space G. In the above, $\boldsymbol{F}(\boldsymbol{U})$ is the flux and $\boldsymbol{Q}(\boldsymbol{U})$ is a relaxation term. The parameter $\varepsilon > 0$ can be interpreted as a characteristic time-scale of the relaxation process.

Systems of the form (1.1) are useful in describing non-equilibrium processes, and therefore have a large number of applications in the physical modeling of different phenomena. Important examples include traffic flow [1], kinetic theory [4] and gas flow in local thermal non-equilibrium [11,7].

A crucial concept for hyperbolic relaxation systems is that of local equilibrium. The equilibrium manifold is defined as

$$M = \{\boldsymbol{U} \in G : \boldsymbol{Q}(\boldsymbol{U}) = 0\}. \tag{1.2}$$

Moreover, the dynamics of the local equilibrium approximation will in general be described through a system of $n \leq N$ conservation laws [5]

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}) = 0 \tag{1.3}$$

for some reduced variable $\boldsymbol{u} = \boldsymbol{u}(x,t)$. We assume that every $\boldsymbol{u}$ uniquely defines an equilibrium state $\mathcal{E}(\boldsymbol{u}) \in M$.

The stability of the relaxation system is intimately connected to the *sub-characteristic condition*, a concept introduced in the linear case by Whitham [18] and later for non-linear 2×2 systems by Liu [13]. The condition states that the wave velocities of the local equilibrium approximation (1.3) should be interlaced in the characteristic wave velocities of the *homogeneous* relaxation system ($\varepsilon \to \infty$). This concept was further developed for $N \times N$ systems by Chen et al. [5] and shown to be directly related to the convexity of the entropy density of the relaxation system.

Since the pioneering work of Liu [13], the study of 2×2 systems has been an important sandbox for investigating the properties of hyperbolic relaxation systems [6,12,14,10]. This approach can be fruitful because 2×2 systems contain much of the same elements of complexity as a general system, while being less cumbersome to work with. Another important approach is the analysis of linearized relaxation systems. Herein, a notable contribution was made by Yong [19,20], who derived stability criteria based on the structure of such relaxation systems. Also, in a recent work by Barker et al. [2], the dynamics of the solution of the St. Venant equations was investigated by studying the dispersion relation of the corresponding linearized system.

For well-behaved relaxation systems it is expected that the solutions of the relaxation system will approach that of the local equilibrium approximation in the *zero relaxation limit* ($\varepsilon \to 0$) [15,5]. If both the homogeneous relaxation system and the conservation law of the local equilibrium approximation are hyperbolic, then they each describe well-defined wave-dynamics. This implies that the magnitude of the relaxation term in general influences both the strength and speed of the waves of the relaxation system.

The purpose of this paper is to investigate the dispersive wave-dynamics of hyperbolic relaxation systems by studying linearized 2×2 systems. The present approach is similar to that of Yong [20], who used linear analysis to investigate the stability of hyperbolic relaxation systems. However, in this work we wish to focus more on the qualitative aspects of the wave-dynamics of the relaxation system, and in particular how this dynamics relates to the magnitude of the relaxation term. By studying the dispersion relation of linearized 2×2 systems, we hope to illuminate some aspects of the transition between the zero relaxation limit ($\varepsilon \to 0$) and the homogeneous ($\varepsilon \to \infty$) limit. For the models we consider in this paper, the zero relaxation limit will coincide with the local equilibrium approximation. Hence, in the following, we will refer to the model obtained in the limit $\varepsilon \to 0$ as the *equilibrium* model and the corresponding limit $\varepsilon \to \infty$ as the *homogeneous* model.

The main contribution of this paper is the discussion of the wave dynamics of the relaxation system in the transitional regime between these two limits. This transitional regime may be characterized by the *stiffness parameter*

$$\varphi \equiv \xi\varepsilon, \tag{1.4}$$

where ξ is the wave number. For 2×2 relaxation systems, we identify a transition point $\hat{\varphi}$ where the wave dynamics changes character from being dominated by the equilibrium dynamics into behaving more like the homogeneous approximation. Moreover, we show that the wave velocities of the 2×2 relaxation system will be monotonic functions of φ. This observation extends the notion of the sub-characteristic condition to the transitional regime; consequently, we refer to this generalization as the *transitional* sub-characteristic condition.

This paper is organized as follows: In Section 2 we give an introduction to 2×2 hyperbolic relaxation systems and their linearized form. We also outline and discuss the assumptions made in this paper regarding the structure of the relaxation term. We introduce two parameters β and γ that characterize the dynamical behavior of such general systems, and provide an interpretation of these parameters in terms of the wave velocities of the homogeneous and equilibrium models.

In Section 3 we calculate the dispersion relation of the linearized 2×2 relaxation system and show that the limiting behavior ($\varepsilon \to 0$ and $\varepsilon \to \infty$) is as expected. We then identify a value of the stiffness parameter φ that may be associated with a point of transition between the equilibrium and homogeneous regimes.

In Section 3.5, we extend Liu's classic notion of the sub-characteristic condition for 2×2 systems [13] into the transitional regime. In particular, we show that any reduction in the stiffness parameter causes the transitional wave velocities to approach each other.

Finally, in Section 4, the paper is summarized and the main conclusions are presented.

2. 2 × 2 hyperbolic relaxation systems

For the calculations in this work, we limit ourselves to 2×2 systems in one spatial dimension, written in the general form

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{1}{\varepsilon}\boldsymbol{Q}(\boldsymbol{U}), \tag{2.1}$$

where $\boldsymbol{U} \in G \subseteq \mathbb{R}^2$ for some state space G. Moreover, we assume that the system (2.1) is hyperbolic in the strict sense, i.e. the Jacobian matrix

$$A(\boldsymbol{U}) = \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}} \tag{2.2}$$

is diagonalizable with real and distinct eigenvalues for all $\boldsymbol{U} \in G$. The eigenvalues of A are then the characteristic speeds of the *homogeneous* relaxation system, seen as the limit $\varepsilon \to \infty$.

2.1. *Linearization*

In the usual way, we consider a linearization of the system (2.1) around a constant equilibrium state. The linear system is given by

$$\partial_t \boldsymbol{U} + A\,\partial_x \boldsymbol{U} = \frac{1}{\varepsilon} R\boldsymbol{U}, \tag{2.3}$$

where A and R are both 2×2 matrices with constant real coefficients.

By denoting

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2.4}$$

we can write the characteristic speeds of the linear system as

$$\mu^{\pm} = \frac{1}{2}(a_{11} + a_{22}) \pm \left(\frac{1}{4}(a_{11} + a_{22})^2 - a_{11}a_{22} + a_{12}a_{21} \right)^{1/2}. \tag{2.5}$$

The assumption of strict hyperbolicity is then given explicitly as

$$\frac{1}{4}\left(a_{11} + a_{22}\right)^2 - a_{11}a_{22} + a_{12}a_{21} > 0. \tag{2.6}$$

2.2. *Structure of the relaxation matrix*

In this work, we will make the following basic assumptions regarding the 2×2 relaxation matrix R:

(i) The matrix R has rank 1

(ii) The matrix R is stable, i.e. it has no eigenvalues with positive real part.

The rationale behind (i) becomes clear when considering the two other possible choices: If the matrix R has rank 0 then it is the zero matrix and there is no relaxation effect on the system. Also, if the matrix has rank 2, the local equilibrium approximation $R\boldsymbol{U} = 0$ will impose two linearly independent constraints on the 2-vector $\boldsymbol{U}$ and the equilibrium approximation will be a constant solution. Since we in this work are interested in the relationship between the dynamics of the relaxation system and that of the local equilibrium approximation, assumption (i) represents the only interesting case.

The condition (ii) is necessary for the solution of the relaxation ODE $\partial_t \boldsymbol{U} = R\boldsymbol{U}$ to converge to a well-defined equilibrium [8]. As observed in [8,19], the relaxation matrix R may be simplified through a variable transformation:

Proposition 2.1. *Under the assumptions (i) and (ii), there exists a change of variables such that a 2×2 relaxation matrix R can be written in the form*

$$R = \begin{bmatrix} 0 & 0 \\ r_{21} & -1 \end{bmatrix}. \tag{2.7}$$

Proof. A 2×2 matrix R that fulfills (i) can, up to a row-swap, be written in the form

$$R = \begin{bmatrix} Kr_{21} & Kr_{22} \\ r_{21} & r_{22} \end{bmatrix}, \qquad K \in \mathbb{R}. \tag{2.8}$$

Therefore, there will exist a matrix

$$T = \begin{bmatrix} 1 & -K \\ 0 & 1 \end{bmatrix} \tag{2.9}$$

representing change of variables $\boldsymbol{U} \to T\boldsymbol{U}$ and a corresponding similarity transform $R \to TRT^{-1}$, yielding a relaxation matrix with zeroes in the first row. We can therefore let

$$R = \begin{bmatrix} 0 & 0 \\ r_{21} & r_{22} \end{bmatrix}, \tag{2.10}$$

by simply assuming that this change of variables already has been performed.

It is straightforward to verify that (ii) for the matrix (2.10) requires $r_{22} < 0$. The absolute value of r_{22} can then be absorbed into the relaxation time ε, yielding the desired form. □

For the purpose of the discussions of this paper, we therefore define the following:

Definition 2.2 (Relaxation system). ***The linear 2 × 2 relaxation system*** *will refer to the general strictly hyperbolic system in the form* (2.3) *where the relaxation matrix R is in the form* (2.7)*.*

2.3. *The local equilibrium approximation*

For the linearized 2×2 system, the equilibrium manifold (1.2) is characterized by $R\boldsymbol{U} = 0$. By denoting $\boldsymbol{U} = [U_1, U_2]^T$, and given the assumed form of the relaxation matrix (2.7), we can write the local equilibrium approximation explicitly as

$$U_2 = r_{21} U_1. \tag{2.11}$$

The dynamics of the reduced variable U_1 is then governed by the advection equation

$$\partial_t U_1 + v^* \partial_x U_1 = 0, \tag{2.12}$$

where the *equilibrium wave velocity* v^* is given by

$$v^* = a_{11} + r_{21} a_{12}. \tag{2.13}$$

2.3.1. *The sub-characteristic condition*

The sub-characteristic condition requires that the wave velocities of the local equilibrium approximation are interlaced in the wave velocities of the hyperbolic relaxation system [5]. For the 2×2 system, this reduces to a simple inequality, as formulated by Liu [13].

Definition 2.3 (The sub-characteristic condition). *Consider the general* 2×2 *relaxation system in the form* (2.1)*. Let* $\mu^{\pm}$ *be the wave velocities of the homogeneous model, and let* v^* *be the wave velocity of the equilibrium model. The sub-characteristic condition states that these velocities must satisfy*

$$\mu^{-} \le v^* \le \mu^{+}. \tag{2.14}$$

For the linear 2×2 relaxation system, the sub-characteristic condition thus takes the form

$$\gamma - \beta^2 \ge 0, \tag{2.15}$$

where we have introduced the convenient shorthands

$$\gamma \equiv \frac{1}{4}\left(a_{11} + a_{22}\right)^2 - a_{11}a_{22} + a_{12}a_{21} \tag{2.16}$$

and

$$\beta \equiv a_{11} + a_{12}r_{21} - \frac{1}{2}\left(a_{11} + a_{22}\right). \tag{2.17}$$

Remark 2.4. In terms of the shorthand (2.16), the condition of strict hyperbolicity of the relaxation system (2.6) can be written in the simple form

$$\gamma > 0. \tag{2.18}$$

The sub-characteristic condition is an important topic in the field of hyperbolic relaxation systems [5,3,7]. As discussed by Natalini [16], it can be seen as a causality principle; the assumption of local equilibrium cannot cause waves to propagate faster than in the full relaxation system. Also, it can be shown that for certain relaxation systems, the sub-characteristic condition is sufficient for the local equilibrium approximation to be the limit of the relaxation system as $\varepsilon \to 0$ [15].

2.4. *An example model*

As an illustrative example, we consider a specific linear 2×2 relaxation system. It was introduced by Jin and Xin [9], and has since been commonly used as an example model [16,10]. The system is given by

$$\partial_t U_1 + \partial_x U_2 = 0 \tag{2.19a}$$

$$\partial_t U_2 + \lambda_R^2 \partial_x U_1 = \frac{1}{\varepsilon}(\lambda_E U_1 - U_2), \tag{2.19b}$$

where λ_E and λ_R are parameters of the model; $\mu^\pm = \pm\lambda_R$ are the characteristic speeds of the homogeneous relaxation system while $v^* = \lambda_E$ is the equilibrium advection speed.

In the context of the general 2×2 system considered in this work, the example model is given by $\gamma = \lambda_R^2$ and $\beta = \lambda_E$. The system is therefore hyperbolic by construction and the sub-characteristic condition can be written as $\lambda_R^2 \geq \lambda_E^2$.

3. Wave-dynamics

The homogeneous relaxation system and the local equilibrium approximation both have well-defined characteristic wave velocities. Moreover, the number of waves in these cases are in general different. The magnitude of the relaxation term will therefore influence both the strength of the waves as well as their respective wave velocities. In this section we seek to illuminate this mechanism by investigating the wave-dynamics of the relaxation system in detail.

3.1. *Plane-wave solutions*

As discussed by Yong [20], there exists for an initial condition $\boldsymbol{U}(x, 0) \in \mathrm{L}^2$ a unique solution to (2.3) in the general form

$$\boldsymbol{U}(x,t) = \sum_\xi \boldsymbol{U}_\xi(x,t) = \sum_\xi \exp\left(H(\xi)\,t\right) \exp\left(i\xi x\right) \hat{\boldsymbol{U}}(\xi). \tag{3.1}$$

Herein, ξ is the wave number and $H(\xi)$ is a 2×2 matrix given by

$$H(\xi) = \frac{1}{\varepsilon} R - i\xi A = \frac{1}{\varepsilon} \begin{bmatrix} -i\varepsilon\xi a_{11} & -i\varepsilon\xi a_{12} \\ r_{21} - i\varepsilon\xi a_{21} & -1 - i\varepsilon\xi a_{22} \end{bmatrix}. \tag{3.2}$$

If $H(\xi)$ is diagonalizable, it can be written in the form

$$H(\xi) = P \begin{bmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{bmatrix} P^{-1}, \tag{3.3}$$

where P is the matrix consisting of the eigenvectors of $H(\xi)$. By using (3.3), we can write the general solution in terms of plane waves as

$$\begin{aligned} \boldsymbol{U}(x,t) = \sum_\xi \Big[& \boldsymbol{U}^+(\xi) \exp\left(i\left(\xi x + \mathrm{Im}\lambda^+\, t\right)\right) \exp\left(\mathrm{Re}\lambda^+\, t\right) \\ & + \boldsymbol{U}^-(\xi) \exp\left(i\left(\xi x + \mathrm{Im}\lambda^-\, t\right)\right) \exp\left(\mathrm{Re}\lambda^-\, t\right) \Big], \end{aligned} \tag{3.4}$$

for some amplitudes $\boldsymbol{U}^\pm(\xi)$.

From the general solution (3.4), we can deduce that associated with each of the two eigenvalues of $H(\xi)$ there is a plane wave with wave-number ξ. For each eigenvalue $\lambda^\pm$, the real part $\mathrm{Re}\lambda^\pm$ is an amplification while the negative imaginary part $-\text{Im}\lambda^{\pm}$ represents a dispersion relation $\omega^{\pm}(\xi)$. Given a dispersion relation, we can obtain the corresponding wave velocity v using the standard relation

$$v^{\pm}(\xi) = \frac{1}{\xi}\omega^{\pm}(\xi) = -\frac{1}{\xi}\text{Im}\lambda^{\pm}. \tag{3.5}$$

By straightforward calculation, the eigenvalues of (3.2) are given by

$$\lambda^{\pm} = \frac{\xi}{2\varphi}\left[-1 - i\varphi\left(a_{11} + a_{22}\right) \pm \left(1 - 4\varphi^2\gamma - i4\varphi\beta\right)^{1/2}\right], \tag{3.6}$$

where we have employed the stiffness parameter φ as given by (1.4).

Introducing the shorthand

$$J = \sqrt{(1 - 4\varphi^2\gamma)^2 + 16\varphi^2\beta^2}, \tag{3.7}$$

we may write the real part of (3.6) as

$$\text{Re}\lambda^{\pm} = \frac{\xi}{2\varphi}\left[-1 \pm \frac{1}{\sqrt{2}}\left(J + 1 - 4\varphi^2\gamma\right)^{1/2}\right]. \tag{3.8}$$

As previously discussed, for hyperbolic relaxation systems the sub-characteristic condition is intimately connected to the stability of the solution. For 2×2 systems the connection can be made explicit, and we here restate for linear systems the following result, which was established for non-linear systems by Chen et al. [5]:

Proposition 3.1. *For linear 2×2 systems as described in Definition 2.2, the linear stability of the solution is equivalent to the sub-characteristic condition.*

Proof. Linear stability requires

$$\text{Re}\lambda^{\pm} \leq 0. \tag{3.9}$$

Inserting (3.8) into (3.9) yields

$$\left(\left(1 - 4\varphi^2\gamma\right)^2 + 16\varphi^2\beta^2\right)^{1/2} + 1 - 4\varphi^2\gamma \leq 2. \tag{3.10}$$

Rearranging and squaring yields

$$\left(1 - 4\varphi^2\gamma\right)^2 + 16\varphi^2\beta^2 \leq \left(1 + 4\varphi^2\gamma\right)^2. \tag{3.11}$$

Furthermore, by canceling terms and rearranging, (3.11) can be simplified to

$$\gamma - \beta^2 \geq 0, \tag{3.12}$$

which is the sub-characteristic condition. □

For the imaginary part of (3.6), we must consider two cases:

(1) The degenerate case $\beta = 0$:

$$\text{Im}\lambda^{\pm} = \begin{cases} -\frac{\xi}{2}\left(a_{11} + a_{22}\right) & \text{if} \quad \varphi \leq \frac{1}{2}\gamma^{-1/2} \\ -\frac{\xi}{2}\left(a_{11} + a_{22}\right) \pm \frac{\xi}{2\varphi}\left(4\varphi^2\gamma - 1\right)^{1/2} & \text{otherwise} \end{cases} \tag{3.13}$$

(2) The non-degenerate case $\beta \neq 0$:

$$\mathrm{Im}\lambda^{\pm} = -\xi \left[\frac{1}{2}(a_{11} + a_{22}) \pm \frac{\mathrm{sgn}(\beta)}{2\sqrt{2}\varphi} \left(J - 1 + 4\varphi^2\gamma \right)^{1/2} \right]. \tag{3.14}$$

The expressions (3.8), (3.13) and (3.14) then completely describe the wave-dynamics of the 2×2 hyperbolic relaxation system.

The wave velocities $v^{\pm}(\varphi)$ of the solution are obtained by applying (3.5) to (3.13) and (3.14), which yields:

(1) The degenerate case $\beta = 0$:

$$v^{\pm}(\varphi) = \begin{cases} \frac{1}{2}(a_{11} + a_{22}) & \text{if} \quad \varphi \leq \frac{1}{2}\gamma^{-1/2} \\ \frac{1}{2}(a_{11} + a_{22}) \mp \frac{1}{2\varphi}\left(4\varphi^2\gamma - 1\right)^{1/2} & \text{otherwise} \end{cases} \tag{3.15}$$

(2) The non-degenerate case $\beta \neq 0$:

$$v^{\pm}(\varphi) = \frac{1}{2}(a_{11} + a_{22}) \pm \frac{\mathrm{sgn}(\beta)}{2\sqrt{2}\varphi} \left(J - 1 + 4\varphi^2\gamma \right)^{1/2}. \tag{3.16}$$

By simple inspection of (3.15)–(3.16), we can immediately deduce some important qualitative aspects of the wave dynamics of the solution. Firstly, the wave velocities of the relaxation system only depend on the variable $\varphi = \varepsilon\xi$. This implies that, as far as the wave velocities are concerned, the zero relaxation limit ($\varepsilon \to 0$) is indistinguishable from the short wave-number limit ($\xi \to 0$). Secondly, by comparing the wave velocities of the relaxation system with the characteristics (2.5) of the homogeneous relaxation system, we can conclude that both these pairs of wave velocities are symmetric around the same root center $(1/2)(a_{11} + a_{22})$. Lastly, the order of the wave velocities in the non-degenerate case (3.16) depend on the sign of the parameter β. From (2.17) it is easy to see that the sign of β is determined by the magnitude of the equilibrium wave velocity v^* relative to the root center $(1/2)(a_{11} + a_{22})$. As will be shown in the following, this choice of ordering lets us associate the λ^+-wave with the equilibrium wave in the stiff limit.

3.2. *Limit behavior*

We now wish to verify that the limiting behavior of the wave velocities and amplification of the plane-waves (3.4) of the general solution. Since the fundamental variable of the wave velocities is φ, we will investigate what we refer to as the stiff ($\varphi \to 0$) and the non-stiff ($\varphi \to \infty$) limit. The following result can be shown by straightforward calculations:

Proposition 3.2 (Non-stiff limit). *In the non-stiff limit the amplifications* (3.8) *are*

$$\lim_{\varphi \to \infty} \mathrm{Re}\lambda^{\pm} = 0, \tag{3.17}$$

and the corresponding wave velocities (3.15)–(3.16) *are*

(1) The degenerate case $\beta = 0$*:*

$$\lim_{\varphi\to\infty} v^{\pm}(\varphi) = \frac{1}{2}(a_{11} + a_{22}) \mp \gamma^{1/2}. \tag{3.18}$$

(2) The non-degenerate case $\beta \neq 0$*:*

$$\lim_{\varphi\to\infty} v^{\pm}(\varphi) = \frac{1}{2}(a_{11} + a_{22}) \pm \mathrm{sgn}(\beta)\gamma^{1/2}. \tag{3.19}$$

Since the non-stiff limit is simply the limit where the magnitude of the relaxation term vanishes, Proposition 3.2 is as expected. The wave-dynamics is equal to that of the homogeneous relaxation system, with wave velocities equal to the characteristic speeds of the flux term.

Perhaps more interesting is the stiff limit. As discussed, in the zero relaxation limit ($\varepsilon \to 0$) the solutions of relaxation systems tend to approach the solution of the local equilibrium approximation [5,15]. This is consistent with the interpretation of ε as a characteristic time-scale of the relaxation—the limit $\varepsilon \to 0$ represents an infinite relaxation speed. The following can be shown by straightforward calculations:

Proposition 3.3 (Stiff limit). *In the stiff limit the amplifications* (3.8) *are*

$$\lim_{\varphi\to 0} \mathrm{Re}\lambda^{\pm} = \lim_{\varphi\to 0} \frac{\xi}{2\varphi}\left(-1 \pm 1\right), \tag{3.20}$$

and the corresponding wave velocities (3.15)–(3.16) *are*

$$\lim_{\varphi\to 0} v^{\pm}(\varphi) = a_{11}\left(\frac{1}{2} \pm \frac{1}{2}\right) + a_{22}\left(\frac{1}{2} \mp \frac{1}{2}\right) \pm a_{12}r_{21}. \tag{3.21}$$

Proposition 3.3 reveals how the 2-wave dynamics of the 2×2 relaxation system approaches the 1-wave dynamics of the local equilibrium approximation as $\varphi \to 0$. The wave velocities of the relaxation system are mirrors of each other around $(1/2)(a_{11} + a_{22})$ for all φ. When $\varphi \to 0$, the wave with wave velocity closest to the equilibrium velocity v^* will be undamped, while the mirror wave will diminish. The dependence of the wave velocities on $\mathrm{sgn}(\beta)$ is such that it is v^+ that is closest to v^*.

3.3. *The degenerate case*

The main purpose of this paper is to investigate the *transitional* ($\varphi \in \langle 0, \infty\rangle$) wave-dynamics of the 2×2 relaxation system. To this end, we first consider the degenerate case $\beta = 0$. For this case, the equilibrium wave-speed is equal to the root center of the wave velocities of the homogeneous relaxation system.

3.3.1. *Wave attenuation*

We may write the amplification factors (3.8) as

$$f^{\pm}(\varphi) \equiv \mathrm{Re}\lambda^{\pm} = \frac{\xi}{2\varphi}\left(-1 \pm \frac{1}{\sqrt{2}}\sqrt{J + 1 - 4\varphi^2\gamma}\right), \tag{3.22}$$

where J is given by (3.7). By assuming $\beta = 0$ and defining $\hat{\varphi}$ as

$$\hat{\varphi} \equiv \frac{1}{2}\gamma^{-1/2}, \tag{3.23}$$

we may write (3.22) as

$$f^{\pm}(\varphi) = \begin{cases} \dfrac{\xi}{2\varphi}\left(\pm\sqrt{1-4\varphi^2\gamma}-1\right) & \text{if } \varphi < \hat{\varphi}, \\ -\dfrac{\xi}{2\varphi} & \text{otherwise.} \end{cases} \tag{3.24}$$

We obtain

$$\frac{\mathrm{d}f^{\pm}}{\mathrm{d}\varphi} = \begin{cases} \dfrac{\xi}{2\varphi^2}\left(1 \mp \left(1-4\varphi^2\gamma\right)^{-1/2}\right) & \text{if } \varphi < \hat{\varphi}, \\ \dfrac{\xi}{2\varphi^2} & \text{if } \varphi > \hat{\varphi}. \end{cases} \tag{3.25}$$

Observe in particular that $f^{\pm}$ is not differentiable at the point $\varphi = \hat{\varphi}$, and that the one-sided limit satisfies

$$\lim_{\varphi \to \hat{\varphi}^-} \frac{\mathrm{d}f^{\pm}}{\mathrm{d}\varphi} = \mp\infty. \tag{3.26}$$

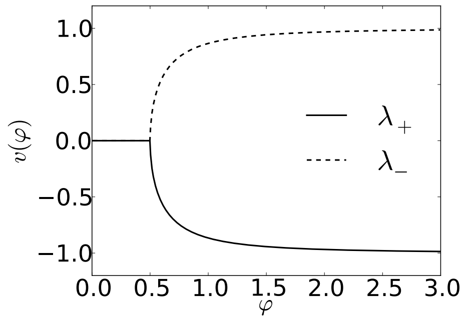
3.3.2. *Summary for the degenerate case*

From the analysis of the degenerate case, we may now conclude the following:

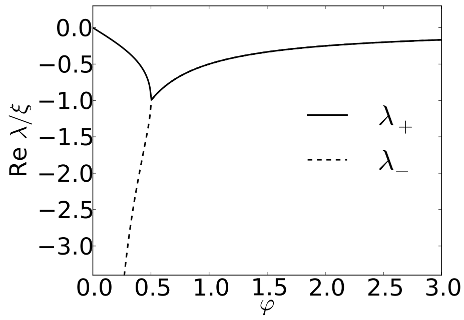
- In the region $\varphi \in \langle\hat{\varphi}, \infty\rangle$, the system displays the 2-wave dynamics of the relaxation system with dampening; both waves are equally attenuated.
- At the point $\varphi = \hat{\varphi}$, there is an abrupt bifurcation of the amplification factors, leading to a strong local reduction of the attenuation of the λ^+-wave, as well as a similar increase in the attenuation of the λ^--wave. Also, at this point, the wave velocities of the relaxation system become equal to the velocity of the local equilibrium approximation.
- In the region $\varphi \in \langle 0, \hat{\varphi}\rangle$, this separation of the waves increases, leading eventually to the λ^--wave being fully suppressed and the λ^+-wave reducing to the non-attenuated wave of the equilibrium system. In this region the wave velocities of the relaxation system are both equal to the wave-speed of the local equilibrium approximation.

Consequently, it makes some sense to interpret the point $\varphi = \hat{\varphi}$ as a *point of transition* where the system changes character from the 2-wave dynamics of the homogeneous relaxation system to being dominated by the 1-wave dynamics of the local equilibrium approximation.

Figure 1 shows the wave velocities and amplifications for the example model (2.19a)–(2.19b) using $\lambda_R = \gamma^{1/2} = 1.0$ and $\lambda_E = \beta = 0.0$. The plot clearly shows a bifurcation at the point $\varphi = \hat{\varphi} = 0.5$. This supports the interpretation of $\hat{\varphi}$ as a point of transition between the two regimes.



(a) Wave velocities



(b) Amplifications

Fig. 1. Wave velocities (3.15) and amplifications (3.8) for the example model (2.19a)–(2.19b), using $\lambda_R = 1$ and $\lambda_E = 0.0$, representing the degenerate case.

3.4. *The non-degenerate case*

We now investigate how the interpretation of $\hat{\varphi}$ as a critical point of transition carries over to the non-degenerate case, given by $\beta \neq 0$.

3.4.1. *Wave attenuation*

For $\beta \neq 0$, the derivative of (3.22) becomes

$$\frac{\mathrm{d}f^\pm}{\mathrm{d}\varphi} = \frac{\xi}{2\varphi^2}\left(1 \mp \frac{1}{\sqrt{2}}\sqrt{J+1-4\varphi^2\gamma}\right) \pm \frac{\xi\gamma(-1+4\varphi^2\gamma+2\frac{\beta^2}{\gamma}-J)}{J\sqrt{\frac{1}{2}\left(J+1-4\varphi^2\gamma\right)}}. \tag{3.27}$$

In this case, the derivatives exist for all $\varphi \in \langle 0, \infty\rangle$. We are now interested in finding any critical points where $f'(\varphi) = 0$. To this end, it will prove convenient to introduce the auxiliary variable Q:

$$Q = \sqrt{\frac{1}{2}\left(J+1-4\varphi^2\gamma\right)}, \tag{3.28}$$

from which φ^2 may be uniquely determined:

$$\varphi^2 = Q^2\frac{(Q+1)(Q-1)}{4(\beta^2 - Q^2\gamma)}. \tag{3.29}$$

Remark 3.4. Observe that when

$$\gamma - \beta^2 = 0, \tag{3.30}$$

the amplification factors reduce to

$$f^+(\varphi) = 0, \tag{3.31}$$

$$f^-(\varphi) = -\frac{\xi}{\varphi}. \tag{3.32}$$

In this case, there is no transitional attenuation of the λ^+-wave, and

$$\frac{\mathrm{d}f^+}{\mathrm{d}\varphi} \equiv 0. \tag{3.33}$$

When $\gamma - \beta^2 \neq 0$, the roots of (3.27) satisfy the equivalent equation

$$h^{\pm}(Q) = Q^3 \pm \zeta \left(Q^2 \mp Q - 1\right) = 0, \tag{3.34}$$

where

$$\zeta = \frac{\beta^2}{\gamma} > 0. \tag{3.35}$$

Through elementary analytical techniques, we may establish that h^+ has a unique root corresponding to $\varphi \in \langle 0, \infty \rangle$, whereas h^- has no such root. Further analysis yields the following proposition.

Proposition 3.5. *Consider a linear* 2×2 *relaxation system as described in Definition 2.2 satisfying* $\beta \neq 0$ *and* $\gamma - \beta^2 \neq 0$*. The amplification of the* λ^+*-wave, given by* (3.8)*, has a unique local extremum* φ_{c} *in the interval*

$$\varphi \in \langle 0, \infty \rangle. \tag{3.36}$$

This critical point is given by

$$\varphi_{\mathrm{c}} = \frac{Q_{\mathrm{c}}^{3/2}}{2\beta}, \tag{3.37}$$

where

$$Q_{\mathrm{c}} = \frac{\nu^{1/3}}{6} + \frac{2\zeta + \frac{2}{3}\zeta^2}{\nu^{1/3}} - \frac{\zeta}{3}, \tag{3.38}$$

$$\nu = 108\zeta - 36\zeta^2 - 8\zeta^3 + 12\sqrt{81\zeta^2 - 66\zeta^3 - 15\zeta^4}. \tag{3.39}$$

On the other hand, the amplification of the λ^-*-wave is unconditionally strictly monotonic for* $\varphi \in \langle 0, \infty \rangle$*.*

We may verify that

$$\lim_{\beta \to 0} \varphi_{\mathrm{c}} = \hat{\varphi}, \tag{3.40}$$

as should be expected.

In summary, we may divide the non-degenerate case $\beta \neq 0$ into 3 further sub-cases:

- *The sub-characteristic condition is strictly satisfied, i. e.*

$$\zeta < 1. \tag{3.41}$$

For all $\varphi \in \langle 0, \infty \rangle$, the λ^+-wave is attenuated, with the amplification factor f^+ having a unique minimum at φ_{c}.

- *The sub-characteristic condition is marginally satisfied, i. e.*

$$\zeta = 1. \tag{3.42}$$

For all $\varphi \in \langle 0, \infty \rangle$, the transitional amplification of the λ^+-wave is identically zero.

- *The sub-characteristic condition is not satisfied, i. e.*

$$\zeta > 1, \tag{3.43}$$

For all $\varphi \in \langle 0, \infty \rangle$, the λ^+-wave is amplified, with the amplification factor f^+ having a unique maximum at φ_c.

For all these cases, the λ^--wave is attenuated with the amplification factor f^- being strictly monotonically increasing.

3.4.2. *An heuristic interpretation*

The analysis of this section indicates that the conclusions drawn from the degenerate case $\beta = 0$ qualitatively carry over to the general case $\beta \neq 0$. In particular, the analysis justifies associating the point

$$\hat{\varphi} = \frac{1}{2}\gamma^{-1/2} \tag{3.44}$$

with a *point of transition*, where wave numbers corresponding to

$$\varphi < \hat{\varphi} \tag{3.45}$$

display a behavior characteristic of the 1-wave equilibrium system, whereas wave numbers corresponding to

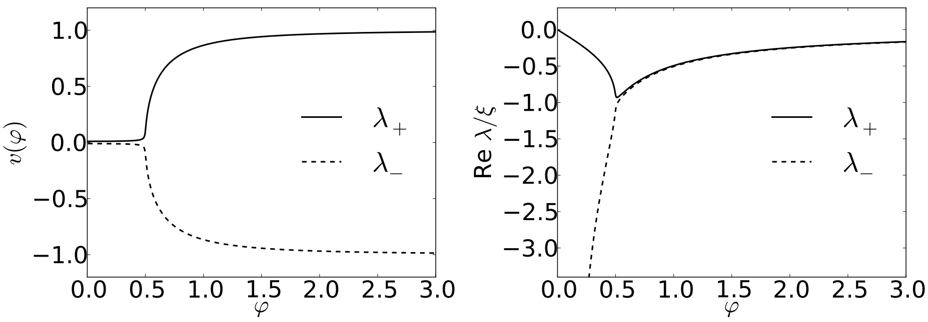
$$\varphi > \hat{\varphi} \tag{3.46}$$

display a behavior more strongly associated with the non-stiff 2-wave relaxation system.

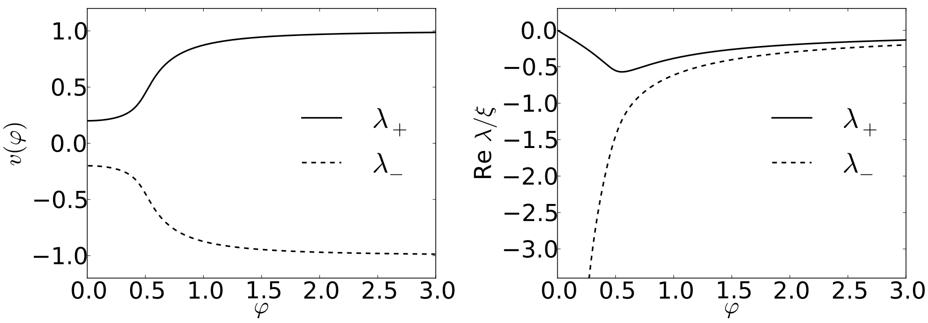
This transition is very obvious in the case $\beta = 0$. As β increases, the transition becomes more smooth while retaining the qualitative behavior. Hence the parameters γ and β may be said to play separate roles in determining the transition between the homogeneous and equilibrium dynamics. Through (3.44), the parameter γ identifies the location of the transition point. The parameter β may be interpreted as a regularization parameter, determining the degree of smoothness of the transition.

Figure 2 shows the wave velocities (3.16) and amplifications (3.8) for the example model, using $\lambda_R = 1.0$ and different values for $\beta = \lambda_E$. The cases considered in Figure 2 all belong to the non-degenerate case. The figure demonstrates that the wave-dynamics approaches that of the degenerate case when β becomes small, and that the transition becomes gradually more smooth with increasing β.

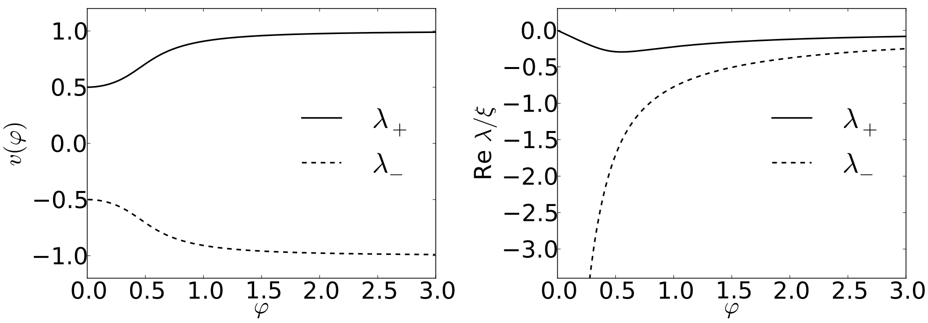
Städtke [17] observed a qualitatively similar behavior as that shown in Figure 2 in his analysis of a 5×5 model for two-phase flow.



(a) $\beta = 0.01$



(b) $\beta = 0.2$



(c) $\beta = 0.5$

Fig. 2. Wave velocities (3.16) and amplifications (3.8) for the example model (2.19a)–(2.19b), using $\lambda_R = 1$ and different $\beta = \lambda_E$.

3.5. *The transitional sub-characteristic condition*

We have observed that the wave velocities of the system are dependent on φ, making the wave dynamics dispersive. The exact nature of this dispersion will be investigated in the following:

Lemma 3.6 (Monotonicity). *For linear* 2×2 *relaxation systems as described* *in Definition 2.2, the transitional Fourier wave velocities* $v^{\pm}(\varphi)$ *will be monotonic functions of* φ, *with*

$$\operatorname{sgn}\left(\frac{\mathrm{d}v^{\pm}(\varphi)}{\mathrm{d}\varphi}\right) = \begin{cases} \pm\operatorname{sgn}(\beta)\operatorname{sgn}(\gamma-\beta^2) & \text{if} \quad \beta \neq 0, \\ 0 & \text{if} \quad \varphi < \hat{\varphi} \quad \text{and} \quad \beta = 0, \\ \mp 1 & \text{if} \quad \varphi > \hat{\varphi} \quad \text{and} \quad \beta = 0. \end{cases} \tag{3.47}$$

Proof. Using (3.16) we can write

$$\frac{\mathrm{d}v^{\pm}(\varphi)}{\mathrm{d}\varphi} = \pm\frac{\operatorname{sgn}(\beta)}{2\sqrt{2}}\frac{1}{\varphi^2}\left(J-1+4\varphi^2\gamma\right)^{-1/2}\left[1-\frac{(1-4\varphi^2\gamma)+8\varphi^2\beta^2}{\left((1-4\varphi^2\gamma)^2+16\varphi^2\beta^2\right)^{1/2}}\right]. \tag{3.48}$$

The absolute value of the second term in the brackets of (3.48) can be written as

$$\left|\frac{(1-4\varphi^2\gamma)+8\varphi^2\beta^2}{\left((1-4\varphi^2\gamma)^2+16\varphi^2\beta^2\right)^{1/2}}\right| = \sqrt{1-\frac{64\varphi^4\beta^2}{(1-4\varphi^2\gamma)^2+16\varphi^2\beta^2}(\gamma-\beta^2)}. \tag{3.49}$$

The cases $\gamma-\beta^2 = 0$ and $\gamma-\beta^2 > 0$ then follow directly. The case $\gamma-\beta^2 < 0$ follows from the fact that the second term in the brackets of (3.48) will be strictly negative with absolute value greater than 1. Differentiating (3.15) completes the proof. □

Lemma 3.6 demonstrates that the higher the wave number, the closer the wave velocities will be to the characteristics of the homogeneous relaxation system. Conversely, components with lower wave numbers will have wave velocities closer to the equilibrium wave-speed and the equilibrium mirror wave-speed. Moreover, since the wave velocities are monotonic in $\varphi = \varepsilon\xi$, for a fixed wave number the wave velocities will also be monotonic in the relaxation time ε.

The monotonicity of the wave velocities, combined with the limiting behavior, gives us the following bounds:

Proposition 3.7. *Consider linear* 2×2 *relaxation systems as described in Definition 2.2. If the sub-characteristic condition is fulfilled, the transitional wave velocities* $v^{\pm}(\varphi)$ *will for all* $\varphi \in \langle 0,\infty\rangle$ *satisfy*

$$\mu^- \le v^-(\varphi) \le a_{22} - a_{12}r_{21} < \frac{1}{2}(a_{11}+a_{22}) < a_{11}+a_{12}r_{21} \le v^+(\varphi) \le \mu^+ \tag{3.50}$$

if $\beta > 0$ *and*

$$\mu^- \le v^+(\varphi) \le a_{11} + a_{12}r_{21} < \frac{1}{2}(a_{11}+a_{22}) < a_{22}-a_{12}r_{21} \le v^-(\varphi) \le \mu^+ \tag{3.51}$$

if $\beta < 0$.

Proof. The result follows directly from the limit behavior from Proposition 3.2 and Proposition 3.3, and the monotonicity from Lemma 3.6. □

As discussed in Section 2.3.1, for 2×2 systems the sub-characteristic condition requires that the single characteristic of the local equilibrium approximation is contained within the two characteristics of the homogeneous relaxation system. Since the wave-dynamics of the relaxation system is dispersive, it has no well defined characteristics. Instead, the wave velocities of the Fourier components of the solution depend on the wave number. In order to generalize the notion of the sub-characteristic condition to the transitional regime, we emphasize the following result:

Proposition 3.8 (Transitional sub-characteristic condition). *Consider linear 2×2 relaxation systems as described in Definition 2.2. If the sub-characteristic condition is fulfilled, the transitional wave velocities $v^{\pm}(\varphi)$ will satisfy*

$$\min\left(v^-(\varphi_2), v^+(\varphi_2)\right) \leq v^{\pm}(\varphi_1) \leq \max\left(v^-(\varphi_2), v^+(\varphi_2)\right), \tag{3.52}$$

for all $\varphi_1, \varphi_2 \in \langle 0, \infty \rangle$ where $\varphi_1 < \varphi_2$.

Proof. This result follows directly from the monotonicity properties of Lemma 1 and the definitions (3.15)–(3.16). □

Note that in the limit $\varphi_1 \to 0$, $\varphi_2 \to \infty$, the transitional sub-characteristic condition reduces to the classical sub-characteristic condition of Definition 2.3, where the equilibrium velocity is given by

$$v^* = \lim_{\varphi \to 0} v^+(\varphi). \tag{3.53}$$

4. Summary

We have investigated the dispersive wave-dynamics of the solutions to 2×2 hyperbolic relaxation systems. By using linear analysis, we have discussed both the limiting and transitional behavior of the wave-dynamics.

Particular attention has been given to the transitional regime, where the wave-dynamics of the relaxation system can be seen as a mix of the dynamics corresponding to the homogeneous relaxation system and that of the local equilibrium system. The wave velocities of the solution to the general 2×2 relaxation system have been shown to be functions of $\varphi = \varepsilon\xi$. This implies that, as far as the wave velocities are concerned, the zero relaxation limit ($\varepsilon \to 0$) is indistinguishable from the low wave number limit. Conversely, the limit $\varepsilon \to \infty$ is indistinguishable from the high wave number limit.

For any such linear relaxation system, we have identified two parameters β and γ that characterize the qualitative behavior in the transitional regime $\varphi \in \langle 0, \infty \rangle$. In particular, these parameters describe a definite transition between the homogeneous relaxation system and the local equilibrium. Herein, the parameter γ determines the location of a transition point, whereas β acts as a "mollifying" parameter for the smoothness of the transition. In the degenerate case $\beta = 0$, an abrupt non-differentiable transition in the wave velocities and amplification occurs in this critical point.

The transitional wave velocities have been shown to be monotonic functions of φ. Combined with the limiting behavior, this implies that if the sub-characteristic condition is fulfilled the wave velocities of the individual Fourier components of the solution will satisfy a transitional sub-characteristic condition. Moreover, because of the way φ is defined, these results all have a dual interpretation, e.g. the wave velocities can be seen as monotonic both in ε and ξ.

The results of this paper have general validity for any 2×2 linear hyperbolic system with a stable relaxation matrix of rank 1. Our results are derived by simple means, yet their main interest lies in the general qualitative insights they provide into such systems. These insights have to the authors' knowledge not been given much attention in the literature.

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