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Linear dependence of bivariate Minimal Support and Locally Refined B-splines over LR-meshes ☆,☆☆

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ABSTRACT

The focus on locally refined spline spaces has grown rapidly in recent years due to the need in Isogeometric Analysis (IgA) of spline spaces with local adaptivity: a property not offered by the strict regular structure of tensor product B-spline spaces. However, this flexibility sometimes results in collections of B-splines spanning the space that are not linearly independent. In this paper we address the minimal number of Minimal Support B-splines (MS B-splines) and of Locally Refined B-splines (LR B-splines) that can form a linear dependence relation. We show that such minimal numbers are six for MS B-splines and eight for LR B-splines. Further results are established to help detecting collections of B-splines that are linearly independent.

1. Introduction

In 2005 Thomas J.R. Hughes et al. (2005) proposed to reconstitute finite element analysis (FEA) within the geometric framework of CAD technologies. This gave rise to Isogeometric Analysis (IgA). It unifies the fields of CAD and FEA by extending the isoparametric concept of the standard finite elements to other shape functions, such as B-splines and non-uniform rational B-splines (NURBS), used in CAD. This does not only allow for an accurate geometrical description, but it also improves smoothness properties. As a consequence, IgA methods often reach a required accuracy using a much smaller number of degrees of freedom (Großmann et al., 2012). Moreover, in some situations, the increased smoothness also improves the stability of the approximations resulting in fewer nonphysical oscillations (Hughes et al., 2005; Manni et al., 2011).

However, in numerical simulations, local (adaptive) refinements are frequently used for balancing accuracy and computational costs. Traditional B-splines and NURBS spaces are formulated as tensor products of univariate B-spline spaces. This means that refining in one of the univariate B-spline spaces will cause the insertion of an entire new row or column of knots in the bivariate spline space, resulting in a global refinement. In order to break the tensor product structure of the underlying mesh, new formulations of multivariate B-splines have been introduced addressing local refineability.

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1.1. Overview of locally refined spline methods

The first local refinement method introduced were the Hierarchical B-splines, or HB-splines (Forsey and Bartels, 1988), whose properties were further analyzed in Kraft (1997). The HB-splines are linearly independent and non-negative. However, partition of unity, which is necessary for the convex hull property (essential for interpreting the B-spline coefficients as control points), was still missing. To rectify this, Truncated Hierarchical B-splines, or THB-splines, were proposed in Giannelli et al. (2012) and further analyzed in Giannelli et al. (2014). In Giannelli et al. (2014) they show how the construction of HB-splines can be modified while preserving the properties of HB-splines, gaining the partition of unity and smaller support of the basis functions.

A different approach, for local refinement, was introduced in Sederberg et al. (2003) with the T-splines. These are defined over *T-meshes*, where T-junctions between axis aligned segments are allowed. T-splines have been used efficiently in CAD applications, being able to produce watertight and locally refined models. However, the use of the most general T-spline concept in IgA is limited by the risk of linear dependence of the resulting splines (Buffa et al., 2010). It is desirable in numerical simulations to use linearly independent basis functions to ensure that the resulting mass and stiffness matrices have full rank and avoid the algorithmic complexity posed by singular matrices. Analysis-Suitable T-splines, or AST-splines, were therefore introduced in da Veiga et al. (2012). As T-splines, AST-splines provide watertight models, obey the convex hull property, and moreover are linearly independent.

There are many other definitions of B-splines over meshes with local refinements, such as PHT-splines (Deng et al., 2008), PB-splines (Engleitner and Jüttler, 2017) and LR B-splines (Dokken et al., 2013). A discussion of the differences and similarities of HB-splines, THB-splines, T-splines, AST-splines and LR B-splines can be found in Dokken et al. (2018).

1.2. LR B-splines and MS B-splines

In this paper we look at Locally Refined B-splines, or LR B-splines, introduced in Dokken et al. (2013). The idea is to extend the knot insertion refinement of univariate B-splines to insertion of local line segments in tensor meshes. The process starts by considering the tensor product B-spline space over a coarse tensor mesh. Then, when a new inserted local line segment divides the support of one or more LR B-splines in two parts, we perform knot insertion to split such B-splines into two (or more) new ones. The final collection of functions does not sum to one in general. However, it is possible to scale them by means of positive weights so that they form a partition of unity; see Dokken et al. (2013, Section 7).

The LR B-splines are a subset of the Minimal Support B-splines, or MS B-splines. As one can guess from their name, MS B-splines are the tensor product B-splines with minimal support, i.e., without superfluous line segments crossing their support, identifiable on the locally refined mesh. The main difference between LR and MS B-splines is that the former ones are defined algorithmically, while the latter are defined by the topology of the mesh.

1.3. Content of the paper

The freedom in the refinement process can result in undesirable collections of LR B-splines. Namely, the LR B-splines obtained at the end of the refinement process may be linearly dependent. Assumptions on the refinement process have to be established in order to ensure linear independence. We start such analysis by looking at conditions on the mesh necessary for linear dependence. We say that functions $\phi_1, \dots, \phi_n : \mathbb{R}^d \to \mathbb{R}$ are **actively linearly dependent** on $\mathbb{R}^d$ if there exist $\alpha_i \in \mathbb{R}$, $\alpha_i \neq 0$ for all $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \alpha_i \phi_i(\boldsymbol{x}) = 0, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Note that we consequentially look at the minimal set of linearly dependent functions by forbidding zero coefficients in the linear combination.

In this work we show that:

- For any bidegree $\boldsymbol{p}$, the minimal number of active MS B-splines in a linear dependence relation is six, while for LR B-splines it is eight.
- These numbers are sharp for any bidegree $\boldsymbol{p} = (p_1, p_2)$ for the MS B-splines and with $p_k \geq 2$, for some $k \in \{1, 2\}$, for the LR B-splines.

We look at the minimal configurations of linear dependence because we conjecture that any linear dependence relation is a refinement of one of these minimal cases. In other words, they are the roots for the linear dependence. In particular, if this is true, by avoiding the minimal cases, the MS B-splines and LR B-splines are always linearly independent and form a basis. Furthermore, to get such lower bounds, we prove results that can be used to understand if the set of B-splines considered is linearly independent or not. In particular, they can be used to improve the Peeling Algorithm (Dokken et al., 2013, Algorithm 6.3) to verify if the LR B-splines defined on a given mesh are linearly independent.

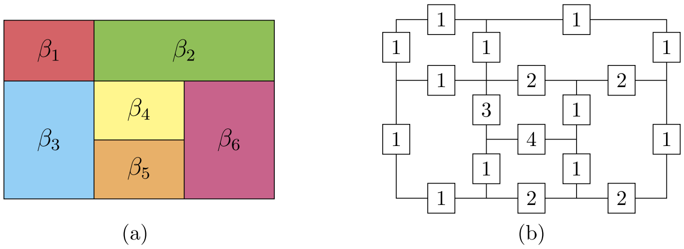

Fig. 1. Example of box–partition and corresponding mesh.

1.4. Structure of the paper

In Section 2 we provide an introduction to the concepts of box-partitions, meshes and LR-meshes. In Section 3 we define the univariate spline space over a knot vector sequence and the bivariate spline space over a box–partition and we recall the dimension formula presented in Pettersen (2013). Then we discuss conditions on the mesh for ensuring that the dimension formula depends only on the topology of the mesh. In Section 4 we recall univariate and bivariate B-splines, their basic properties and the knot insertion procedure. In Section 5 we define the MS B-splines and the LR B-splines and we show when these two sets are different. In Section 6 we study the spanning properties of the LR and MS B-splines. In particular we state necessary and sufficient conditions for spanning the full spline space. Knowing the dimension of the spline space, we can check linear dependencies just by counting the elements in the LR, or MS, B-spline set. In Section 7, we identify necessary features for a linear dependence relation and we derive the minimal number of active MS B-splines needed in a linear dependence relation. In Section 8, we compute the minimal number of active LR B-splines in a linear dependence relation. In Section 9 we recall briefly the Peeling Algorithm for checking linear independence and we show how to improve it by using the results of Section 7. Finally, we summarize the main results and discuss future work in Section 10.

2. Box–partitions and LR–meshes

The purpose of this section is to describe box–partitions in 2D and define bivariate LR–meshes. For our scope, and sake of simplicity, we decided to restrict general definitions, valid in any dimension, to the 2D case; we refer to Dokken et al. (2013) for the general theory.

Definition 2.1. Given an axis-aligned rectangle $\Omega \subseteq \mathbb{R}^2$, a **box-partition** of Ω is a finite collection $\mathcal{E}$ of axis-aligned rectangles in Ω, called **elements**, such that:

1. $\mathring{\beta}_1 \cap \mathring{\beta}_2 = \varnothing$ for any $\beta_1, \beta_2 \in \mathcal{E}$, with $\beta_1 \neq \beta_2$.
2. $\bigcup_{\beta \in \mathcal{E}} \beta = \Omega$.

Definition 2.2. Given a box–partition $\mathcal{E}$, we define the **vertices of** $\mathcal{E}$ as the vertices of its elements. In particular, a vertex of $\mathcal{E}$ is called **T-vertex** if it is the intersection of three elements edges. We denote as $\mathcal{V}$ the set of vertices of $\mathcal{E}$.

Definition 2.3. Given a box–partition $\mathcal{E}$ of a rectangle $\Omega \in \mathbb{R}^2$, a **meshline** of $\mathcal{E}$ is a segment contained in an element edge, connecting two and only two vertices of $\mathcal{V}$ at its end–points. The collection of all the meshlines of the box-partition is called **mesh**, $\mathcal{M}$. Given a mesh $\mathcal{M}$, one can define a multiplicity function $\mu : \mathcal{M} \to \mathbb{N}^*$ that associates a positive integer to every meshline, called **multiplicity of the meshline**. A mesh that has an assigned multiplicity function μ is called **μ-extended mesh**.

When the T-vertices of $\mathcal{E}$ occur only on $\partial\Omega$ and every colinear meshlines have same multiplicity, the corresponding μ-extended mesh is called **tensor mesh**.

Finally, if every meshline of a box-partition $\mathcal{E}$ has the same multiplicity m we say that the corresponding μ-extended mesh has multiplicity m.

In this work we only consider μ-extended meshes. Therefore, we will only write meshes for μ-extended meshes to simplify the notation.

Fig. 1 shows an example of box-partition and associated mesh: in (a) the box-partition $\mathcal{E}$ and in (b) the corresponding mesh $\mathcal{M}$. The meshlines are identified by squares reporting the associated multiplicities.

A meshline can be expressed as the Cartesian product of a point in $\mathbb{R}$ and a finite interval. Let $\alpha \in \mathbb{R}$ be the value of such a point and let $k \in \{1, 2\}$ be its position in the Cartesian product. If $k = 1$ the meshline is vertical and if $k = 2$ the meshline is horizontal. We sometimes write k-meshline to specify the direction of the meshline and (k, α)-meshline to specify exactly on what axis-parallel line in $\mathbb{R}^2$ the meshline lies.

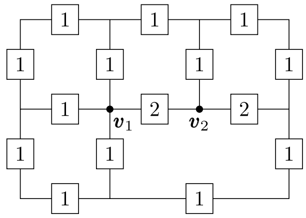

Fig. 2. Example of computation of vertical and horizontal multiplicities.

Definition 2.4. Given a box-partition $\mathcal{E}$ and an axis-aligned segment γ, we say that γ **traverses** $\beta \in \mathcal{E}$ if $\gamma \subseteq \beta$ and the interior of β is divided into two parts by γ, i.e., $\beta \backslash \gamma$ is not connected. A split is a finite union of contiguous and colinear axis-aligned segments $\gamma = \cup_i \gamma_i$ such that every γ_i either is a meshline of the box-partition or γ_i traverses some $\beta \in \mathcal{E}$.

As for meshlines, we sometimes write k-split with $k \in \{1, 2\}$ to specify the direction of the split or (k, α)-split to specify on what axis-parallel line in $\mathbb{R}^2$ the split lies, that is, to specify that it lies on the line $\{(x_1, x_2) : x_k = \alpha\}$.

Definition 2.5. A mesh $\mathcal{M}$ has **constant splits** if any split γ in $\mathcal{M}$ is made of meshlines of the same multiplicity.

When a split γ is inserted in a box–partition $\mathcal{E}$, any traversed $\beta \in \mathcal{E}$ is replaced by the two subrectangles β_1, β_2 given by the closures of the connected components of $\beta \backslash \gamma$. The resulting new box-partition is indicated as $\mathcal{E} + \gamma$ and its corresponding mesh as $\mathcal{M} + \gamma$. Assigned a positive integer μ_γ to γ, the multiplicities of the meshlines in $\mathcal{M} \cap (\mathcal{M} + \gamma)$ not contained in γ are unchanged, while the multiplicities of those that are in γ are increased by μ_γ. The new meshlines contained in $(\mathcal{M} + \gamma) \backslash \mathcal{M}$ have multiplicity equal to μ_γ. If μ was the multiplicity function associated to $\mathcal{M}$, the multiplicity function on the refined mesh $\mathcal{M} + \gamma$ is denoted as $\mu + \mu_\gamma$. The meshes used in applications are often result of a mesh refinement process, that is, given an initial coarse tensor mesh $\mathcal{M}_1$ and a sequence of splits γ_i with associated integers μ_{γ_i} for $i = 1, \dots, N-1$, the meshes considered are the final element of a sequence of meshes of the form $\mathcal{M}_{i+1} = \mathcal{M}_i + \gamma_i$ where the associated multiplicity are $\mu_{i+1} = \mu_i + \mu_{\gamma_i}$. The LR–meshes are a particular subclass of this kind of meshes.

Definition 2.6. An **LR-mesh** is a mesh $\mathcal{M}$ obtained through a sequence of split insertions:

$$\mathcal{M}_1 \text{ is a tensor mesh,}$$
$$\mathcal{M}_{i+1} = \mathcal{M}_i + \gamma_i \text{ has constant splits, for } i = 1, \dots, N-1$$

and $\mathcal{M} = \mathcal{M}_N$, for some N.

In the remaining of this section we introduce the knot vector on a split and the length of it. These concepts will help us to analyze the spanning properties of the LR B-splines and the increase in the spline space dimension due to a mesh refinement.

Definition 2.7. Given a mesh $\mathcal{M}$ corresponding to a box-partition $\mathcal{E}$, for any vertex $\boldsymbol{v}$ in $\mathcal{V}$ we define

$$\mu_1(\boldsymbol{v}) = \max\{\mu(\gamma) : \boldsymbol{v} \in \gamma \text{ and } \gamma \text{ 1-meshline of } \mathcal{M}\}$$
$$\mu_2(\boldsymbol{v}) = \max\{\mu(\gamma) : \boldsymbol{v} \in \gamma \text{ and } \gamma \text{ 2-meshline of } \mathcal{M}\}$$

$\mu_1(\boldsymbol{v})$ is called **vertical multiplicity** and $\mu_2(\boldsymbol{v})$ **horizontal multiplicity** of vertex $\boldsymbol{v}$.

In Fig. 2 is reported an example of computation of horizontal and vertical multiplicities for two vertices of a box–partition. The meshlines on the left- and right-hand side of $\boldsymbol{v}_1$ have multiplicity 1 and 2 respectively. So $\mu_2(\boldsymbol{v}_1) = \max\{1, 2\} = 2$. The meshlines above and below $\boldsymbol{v}_1$ have both multiplicity 1, so that $\mu_1(\boldsymbol{v}_1) = 1$. Concerning $\boldsymbol{v}_2$, we have $\mu_2(\boldsymbol{v}_2) = 2$, whereas $\mu_1(\boldsymbol{v}_2) = \max\{1\} = 1$ because there is no meshline below $\boldsymbol{v}_2$.

Definition 2.8. Given a (k, α)-split γ in a mesh $\mathcal{M}$, all the vertices where γ intersects the meshlines of $\mathcal{M}$, orthogonal to it, have kth-coordinate equal to α and different $(3-k)$th coordinate. We define the **knot vector on** γ as the increasing sequence $\boldsymbol{\tau} \subseteq \mathbb{R}$ given by such $(3-k)$th coordinates. The elements of such sequence are called **knots**. We further define the multiplicity function of the knot vector as the μ_{3-k} multiplicity function of the corresponding vertices. We say that $\boldsymbol{\tau}$ **has length** d if the multiplicities of its knots sum to d.

3. Spline spaces

In this section we define the univariate spline space over a knot vector and the bivariate spline space over a box-partition. In particular we provide the dimension formula of such spaces. For the bivariate space, the formula, introduced in Pettersen (2013), presents terms depending on the size of the box–partition elements. This means that the dimension is unstable, i.e., spline spaces on meshes with the same topology might have a different dimension. Therefore, we recall sufficient conditions for avoiding such terms, making the formula dependent only on the mesh topology.

3.1. Spline space on a knot vector sequence

Definition 3.1. Given an increasing sequence $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ of real numbers, a positive integer p and a function $\mu : \boldsymbol{\tau} \to \mathbb{N}^*$ such that $0 \le \mu(\tau_i) \le p+1$ for all i, we define the corresponding **spline knot vector** as the triple $\boldsymbol{\tau}_p^\mu = (\boldsymbol{\tau}, \mu, p)$.

Given a spline knot vector $\boldsymbol{\tau}_p^\mu$, we say that $\tau_i \in \boldsymbol{\tau}$ has **full multiplicity** if $\mu(\tau_i) = p+1$ and we say that $\boldsymbol{\tau}_p^\mu$ is **open** if τ_1 and τ_n have full multiplicity.

Sometimes it is more convenient to write a spline knot vector, in the equivalent way, as the couple $\boldsymbol{t}_p = (\boldsymbol{t}, p)$ where $\boldsymbol{t}$ is a non–decreasing sequence $\boldsymbol{t} = (t_1, \dots, t_\ell)$, i.e., with $t_i \le t_{i+1}$, where $\ell = \sum_{i=1}^n \mu(\tau_i)$ and

$$\underbrace{t_1 = \ldots = t_{\mu(\tau_1)}}_{=\tau_1} < \underbrace{t_{\mu(\tau_1)+1} = \ldots = t_{\mu(\tau_1)+\mu(\tau_2)}}_{=\tau_2} < \ldots$$

We use bold Greek letters with the multiplicity function in superscript in the first way of expression and bold Latin letters for the second way.

Given a degree p, we denote as $\Pi_p \subset \mathbb{R}[t]$ the vector space spanned by the monomials t^j such that $0 \le j \le p$.

Definition 3.2. Given a spline knot vector $\boldsymbol{\tau}_p^\mu = (\boldsymbol{\tau}, \mu, p)$ with $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$, we define the univariate **spline space on the spline knot vector** $\boldsymbol{\tau}_p^\mu$, denoted $\mathbb{S}(\boldsymbol{\tau}_p^\mu)$, or equivalently $\mathbb{S}(\boldsymbol{t}_p)$, as the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that

1. f is zero outside $[\tau_1, \tau_n]$,
2. the restrictions of f to the intervals $[\tau_i, \tau_{i+1})$ for $i < n-1$ and $[\tau_{n-1}, \tau_n]$ are polynomials in Π_p,
3. f is $C^{p-\mu(\tau_i)}$-continuous at τ_i.

The following is the dimension of the spline space over a knot vector. It is a well-known result, proved, e.g., in Schumaker (2007).

Theorem 3.3. *Given a spline knot vector $\boldsymbol{\tau}_p^\mu = (\boldsymbol{\tau}, \mu, p)$ with $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$, the corresponding spline space $\mathbb{S}(\boldsymbol{\tau}_p^\mu)$ has dimension*

$$\dim \mathbb{S}(\boldsymbol{\tau}_p^\mu) = \max\left\{\sum_{i=1}^n \mu(\tau_i) - (p+1), 0\right\}. \tag{1}$$

Therefore, if $\boldsymbol{t}_p$ has cardinality $p+r+1$ for some $r \ge 1$, then $\dim \mathbb{S}(\boldsymbol{t}_p) = r$. There are many possible bases for $\mathbb{S}(\boldsymbol{t}_p)$. One possibility is provided by a classical result in spline theory, called Curry-Schoenberg Theorem (de Boor, 1978, Theorem 44). It ensures that the so called B-spline functions of degree p, defined on the knot vector $\boldsymbol{t}_p$, can be used as a possible basis:

$$\mathbb{S}(\boldsymbol{t}_p) = \text{span}\,\{B[\boldsymbol{t}_p^i]\}_{i=1}^r \quad \text{with}\, \boldsymbol{t}_p^i = (t_i, \dots, t_{i+p+1}) \subseteq \boldsymbol{t}_p.$$

For a brief introduction to B-splines we refer to Section 4.

3.2. Spline space on a box–partition

Definition 3.4. A **spline mesh** in $\mathbb{R}^2$ is a triple $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ where $\mathcal{M}$ is a mesh from a box–partition $\mathcal{E}$, $\boldsymbol{p} = (p_1, p_2)$ is a pair of positive integers and $\mu : \mathcal{M} \to \mathbb{N}^*$ is a multiplicity function such that $1 \le \mu(\gamma) \le p_k + 1$ for every k-meshline $\gamma \in \mathcal{M}$. In particular, if a k-meshline γ has multiplicity $p_k + 1$ we say that γ has **full multiplicity** and a spline mesh $\mathcal{N}$ is **open** if every boundary meshline has full multiplicity. A spline mesh $\mathcal{N}$ where $\mathcal{M}$ is an LR-mesh will be called **spline LR-mesh**.

Remark 3.5. Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, one can define a spline knot vector on any k-split of $\mathcal{M}$, for $k \in \{1, 2\}$: the sequence $\boldsymbol{\tau}$ and the multiplicity function μ_{3-k} are described in Definition 2.8 and the degree is p_{3-k}.

Given a bidegree $\boldsymbol{p} = (p_1, p_2)$, we denote as $\Pi_{\boldsymbol{p}} \subset \mathbb{R}[x, y]$ the vector space spanned by the monomials $x^{i_1} y^{i_2}$ such that $0 \le i_k \le p_k$ for $k = 1, 2$.

Definition 3.6. Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ corresponding to a box–partition $\mathcal{E}$ of a rectangle $\Omega = [a_1, b_1] \times [a_2, b_2]$, for any element $\beta \in \mathcal{E}$, $\beta = J_1 \times J_2$ with $J_k = [a_{\beta,k}, b_{\beta,k}]$, we set

$$\tilde{\beta} = \tilde{J}_1 \times \tilde{J}_2 \text{ with } \tilde{J}_k = \begin{cases} [a_{\beta,k}, b_{\beta,k}) & \text{if } b_{\beta,k} < b_k \\ [a_{\beta,k}, b_{\beta,k}] & \text{if } b_{\beta,k} = b_k. \end{cases} \tag{2}$$

The **spline space on** $\mathcal{N}$, denoted by $\mathbb{S}(\mathcal{N})$, is the set of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that

1. f is zero outside Ω,
2. for each element $\beta \in \mathcal{E}$, the restriction of f to $\tilde{\beta}$ is a bivariate polynomial function in $\Pi_{\boldsymbol{p}}$,
3. for each k-meshline $\gamma \in \mathcal{M}$, f is $C^{p_k - \mu(\gamma)}$-continuous across γ.

The general dimension formula of the spline space over spline meshes is presented in Pettersen (2013) and has terms depending on the size of the box–partition elements. This makes the dimension of the spline space unstable (Li and Chen, 2011), i.e., not only dependent on the mesh topology. However, if we consider the spline space over a spline LR-mesh built so that

LR-rule 1 the starting tensor mesh $\mathcal{M}_1$ has at least $p_1 + 2$ vertical splits and $p_2 + 2$ horizontal splits counting their multiplicities,

LR-rule 2 for $k \in \{1, 2\}$, the knot vector on any maximal k-split has length at least $p_{3-k} + 2$ at any step in the construction of the LR-mesh,

then, one can prove, by using the results in Pettersen (2013), that, called $\mathcal{M}^k$ the set of all the k-meshlines in $\mathcal{M}$, for $k \in \{1, 2\}$, and $|\mathcal{E}|$ the cardinality of $\mathcal{E}$, we have

$$\begin{aligned} \dim \mathbb{S}(\mathcal{N}) = & \sum_{\boldsymbol{v} \in \mathcal{V}} [(p_1 - \mu_1(\boldsymbol{v}) + 1)(p_2 - \mu_2(\boldsymbol{v}) + 1)] \\ & - (p_2 + 1) \sum_{\beta \in \mathcal{M}^1} [(p_1 - \mu(\beta) + 1)] - (p_1 + 1) \sum_{\beta \in \mathcal{M}^2} [(p_2 - \mu(\beta) + 1)] \\ & + |\mathcal{E}|(p_1 + 1)(p_2 + 1), \end{aligned} \tag{3}$$

which depends only on the topology of the mesh. In this paper we will always assume the LR-rules for constructing LR-meshes.

Remark 3.7. In the LR-mesh building process, any extension of an older split is allowed being LR-rule 2 satisfied on the new mesh.

From equation (3), it is possible to prove the dimension increasing formula (Dokken et al., 2013, Theorem 5.5). Knowing $\dim \mathbb{S}(\mathcal{N})$, through this formula, one can easily compute the dimension of the spline space on a refined spline mesh $\mathcal{N} + \gamma := (\mathcal{M} + \gamma, \mu + \mu_\gamma, \boldsymbol{p})$. First, we need to introduce the concept of expanded spline knot vector on a split.

Definition 3.8. When a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ is refined by inserting a k-split γ, since it is a split in $\mathcal{M} + \gamma$, γ has a spline knot vector on it, $\boldsymbol{\tau}_{p_{3-k}}^{\mu_{3-k}}$, with assigned multiplicity μ_{3-k}. The **expanded spline knot vector** on γ, $\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}$, has same sequence $\boldsymbol{\tau}$, same degree p_{3-k} and same multiplicity function μ_{3-k} except that, in case γ is an extension of a split of $\mathcal{M}$, it is assigned full multiplicity to the point of $\boldsymbol{\tau}$ corresponding to the joint vertex of the extension.

In particular, if γ is an extension of two splits γ_1, γ_2 in $\mathcal{M}$, i.e., γ is the link between γ_1, γ_2, then the first and last knots in the expanded spline knot vector on γ have full multiplicity.

We can now give the dimension increasing formula.

Theorem 3.9. *Given a spline LR-mesh $\mathcal{N}$ and a new k-split γ such that the expanded spline knot vector $\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}$ on γ has length $p_{3-k} + r + 1$ with $r \ge 1$, then the spline space on the refined spline mesh $\mathcal{N} + \gamma := (\mathcal{M} + \gamma, \mu + \mu_\gamma, \boldsymbol{p})$ has dimension*

$$\dim \mathbb{S}(\mathcal{N} + \gamma) = \dim \mathbb{S}(\mathcal{N}) + \dim \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}) = \dim \mathbb{S}(\mathcal{N}) + r.$$

4. Univariate B-splines and bivariate B-splines

In this section we recall the definition of B-splines and their main properties. In particular, we state the knot insertion algorithm, which is used for the definition of LR B-splines. For a complete overview on B-splines we refer to de Boor (1978) and Schumaker (2007).

4.1. Univariate B-splines

Definition 4.1. For a non-decreasing sequence $\boldsymbol{t} = (t_1, t_2, \dots, t_{p+2})$ we define a **B-spline** $B[\boldsymbol{t}] : \mathbb{R} \to \mathbb{R}$ **of degree** $p \geq 0$ recursively by

$$B[\boldsymbol{t}](t) = \frac{t - t_1}{t_{p+1} - t_1} B[t_1, \dots, t_{p+1}](t) + \frac{t_{p+2} - t}{t_{p+2} - t_2} B[t_2, \dots, t_{p+2}](t), \tag{4}$$

where each time a fraction with zero denominator appears, it is taken as zero. The initial B-splines of degree 0 on $\boldsymbol{t}$ are defined as

$$B[t_i, t_{i+1}](t) := \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1}; \\ 0 & \text{otherwise}; \end{cases} \quad \text{for } i = 1, \dots, p+1. \tag{5}$$

The sequence $\boldsymbol{t}$ is called **knot vector** of $B[\boldsymbol{t}]$ and t_j are its **knots**. A knot t_j has multiplicity $\mu(t_j)$ if it appears $\mu(t_j)$ times in $\boldsymbol{t}$.

Proposition 4.2 *(Properties). Given a degree $p \geq 0$ and a knot vector $\boldsymbol{t} = (t_1, \dots, t_{p+2})$,*

- *supp $B[\boldsymbol{t}] = [t_1, t_{p+2}]$,*
- *$B[\boldsymbol{t}]$ restricted to every nontrivial half-open element $[t_i, t_{i+1})$ is in Π_p,*
- *$B[\boldsymbol{t}]$ is $C^{p-\mu(t_j)}$-continuous at any knot t_j of multiplicity $\mu(t_j)$.*

Theorem 4.3 *(knot insertion). Given a degree p and a knot vector $\boldsymbol{t} = (t_1, \dots, t_{p+2})$, suppose we insert a knot $\hat{t} \in (t_1, t_{p+2})$. We obtain two knot vectors $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$, considering the first and the last $p+2$ knots respectively in $(t_1, \dots, \hat{t}, \dots, t_{p+2})$. Then there exist $\alpha_1, \alpha_2 \in (0, 1]$ such that*

$$B[\boldsymbol{t}] = \alpha_1 B[\boldsymbol{t}_1] + \alpha_2 B[\boldsymbol{t}_2]. \tag{6}$$

4.2. Bivariate B-splines

Definition 4.4. Consider a bidegree $\boldsymbol{p} = (p_1, p_2)$. Let $\boldsymbol{x} = (x_1, \dots, x_{p_1+2})$ and $\boldsymbol{y} = (y_1, \dots, y_{p_2+2})$ be nondecreasing sequences. We define the **tensor product B-spline** $B[\boldsymbol{x}, \boldsymbol{y}] : \mathbb{R}^2 \to \mathbb{R}$ by

$$B[\boldsymbol{x}, \boldsymbol{y}](x, y) := B[\boldsymbol{x}](x) B[\boldsymbol{y}](y), \tag{7}$$

where $B[\boldsymbol{x}]$ and $B[\boldsymbol{y}]$ are the univariate B-splines defined on $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

The pair $\boldsymbol{x}, \boldsymbol{y}$ identifies a tensor mesh in $[x_1, x_{p_1+2}] \times [y_1, y_{p_2+2}]$, $\mathcal{M}[\boldsymbol{x}, \boldsymbol{y}]$. In fact, a knot in the x-direction x_i corresponds to the 1-split

$$\gamma = \bigcup_{j=1}^{p_2+1} \gamma_j \quad \text{with } \gamma_j = \{x_i\} \times [y_j, y_{j+1}],$$

and multiplicity $\mu[\boldsymbol{x}, \boldsymbol{y}](\gamma_j)$ equal to the multiplicity of x_i in $\boldsymbol{x}$, for all j. In the same way the knots y_j in $\boldsymbol{y}$ identify the 2-splits in $\mathcal{M}[\boldsymbol{x}, \boldsymbol{y}]$ and their assigned multiplicities.

The properties of univariate B-splines are conserved by the tensor product B-splines:

- supp $B[\boldsymbol{x}, \boldsymbol{y}] = [x_1, x_{p_1+2}] \times [y_1, y_{p_2+2}]$.
- $B[\boldsymbol{x}, \boldsymbol{y}]$ is a piecewise bivariate polynomial of bidegree $\boldsymbol{p}$.
- $B[\boldsymbol{x}, \boldsymbol{y}]$ is $C^{p_k-\mu(\gamma)}$-continuous across each k-meshline γ of $\mathcal{M}[\boldsymbol{x}, \boldsymbol{y}]$.

As in the univariate case, after the insertion of a knot $\hat{x}$ in $\boldsymbol{x}$, we define $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ considering in $(x_1, \dots, \hat{x}, \dots, x_{p_1+2})$ the first and last $p_1 + 2$ knots respectively and we can write $B[\boldsymbol{x}, \boldsymbol{y}]$ in terms of two B-splines defined on the two new pairs of knot vectors

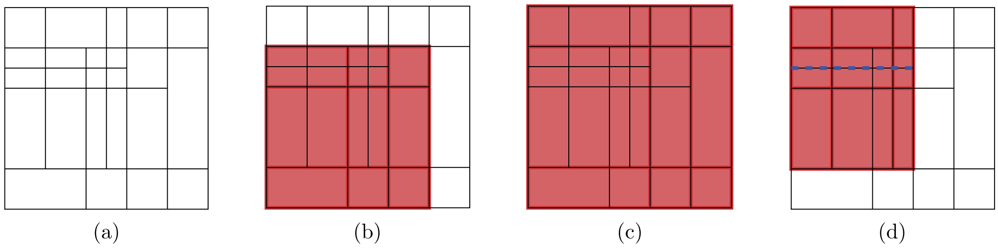

Fig. 3. Support of B-splines of bidegree $(2,2)$ on a mesh $\mathcal{M}$ of multiplicity 1. The mesh is shown in (a). The B-splines whose supports are depicted in (b) and (c) have minimal support on $\mathcal{M}$. The tensor meshes defined by their knots in their supports are highlighted with thicker lines. On the other hand, the B-spline in (d) does not have minimal support on $\mathcal{M}$: the collection of meshlines contained in the dashed line disconnects the support.

$$B[\boldsymbol{x},\boldsymbol{y}] = \alpha_1 B[\boldsymbol{x}_1,\boldsymbol{y}] + \alpha_2 B[\boldsymbol{x}_2,\boldsymbol{y}] \quad \text{with } \alpha_1,\alpha_2 \in (0,1]. \tag{8}$$

The same holds when inserting a knot $\hat{y}$ in $\boldsymbol{y}$.

Finally, consider a spline mesh $\mathcal{N} = (\mathcal{M},\mu,\boldsymbol{p})$ with $\mathcal{M}$ a tensor mesh. Then there exist two spline knot vectors $\boldsymbol{x}_{p_1}$ and $\boldsymbol{y}_{p_2}$ that identify $\mathcal{M}$, $\mathcal{M} = \mathcal{M}[\boldsymbol{x}_{p_1},\boldsymbol{y}_{p_2}]$, as explained before for the tensor mesh $\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ in the support of $B[\boldsymbol{x},\boldsymbol{y}]$. Assume that $\boldsymbol{x}_{p_1}$ and $\boldsymbol{y}_{p_2}$ have length p_1+r_1+1 and p_2+r_2+1 respectively, with $r_1,r_2 \geq 1$. We can apply the Curry-Schoenberg Theorem on each spline knot vector and state that

$$\mathbb{S}(\mathcal{N}) = \operatorname{span}\{B[\boldsymbol{x}^i_{p_1},\boldsymbol{y}^j_{p_2}]\} \quad \text{with } i=1,\dots,r_1 \text{ and } j=1,\dots,r_2,$$

where $\boldsymbol{x}^i_{p_1} = (x_i,\dots,x_{i+p_1+1}) \subseteq \boldsymbol{x}_{p_1}$ and $\boldsymbol{y}^j_{p_2} = (y_j,\dots,y_{j+p_2+1}) \subseteq \boldsymbol{y}_{p_2}$.

5. Minimal Support B-splines and Locally Refined B-splines

In this section we define first the Minimal Support B-splines, or MS B-splines, and then the Locally Refined B-splines, or LR B-splines. As we will see the LR B-splines are created algorithmically, refining, after the insertion of a split in the mesh, the B-splines whose support is traversed by the split through the knot insertion procedure. The main difference with the MS B-splines is that these latter are not always the result of a knot insertion. For a given bidegree, they depend only on the position and multiplicities of the meshlines in the mesh.

Definition 5.1. Given a bivariate B-spline $B[\boldsymbol{x},\boldsymbol{y}]$ and a split γ, we say that γ **traverses** $B[\boldsymbol{x},\boldsymbol{y}]$ if $\operatorname{supp} B[\boldsymbol{x},\boldsymbol{y}]\backslash\gamma$ is not connected.

Definition 5.2. Given a mesh $\mathcal{M}$ and a B-spline $B[\boldsymbol{x},\boldsymbol{y}]$, we say that $B[\boldsymbol{x},\boldsymbol{y}]$ has **support** on $\mathcal{M}$ if the meshlines in $\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ can be obtained as unions of meshlines in $\mathcal{M}$, and their multiplicities in $\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ are less than or equal to the multiplicities of the corresponding meshlines in $\mathcal{M}$. Furthermore, we say that $B[\boldsymbol{x},\boldsymbol{y}]$ has **minimal support** on $\mathcal{M}$ if it has support on $\mathcal{M}$, the multiplicities of the interior meshlines in $\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ are equal to the multiplicities of the corresponding meshlines in $\mathcal{M}$ and there is no split γ in $\mathcal{M}\backslash\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ that traverses $B[\boldsymbol{x},\boldsymbol{y}]$. Given a spline mesh $\mathcal{N} = (\mathcal{M},\mu,\boldsymbol{p})$, the set of the minimal support B-splines, or MS B-splines, on $\mathcal{N}$ of bidegree $\boldsymbol{p}$ is denoted as $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$.

Fig. 3 shows examples of B-splines of bidegree $(2,2)$ with support on a mesh of multiplicity 1. In particular, the B-splines considered in Fig. 3(b)–(c) have minimal support, while the support of the B-spline in Fig. 3(d) can be disconnected by the split γ, visualized by a dashed line in the figure.

Given a mesh $\mathcal{M}$ and a B-spline $B[\boldsymbol{x},\boldsymbol{y}]$ with support in $\mathcal{M}$, assume that there exists a (k,α)–split γ in $\mathcal{M}\backslash\mathcal{M}[\boldsymbol{x},\boldsymbol{y}]$ that traverses $B[\boldsymbol{x},\boldsymbol{y}]$. Assume also that the meshlines in γ have all the same multiplicity m. One could then consider α as an extra knot of multiplicity m in the kth knot vectors of $B[\boldsymbol{x},\boldsymbol{y}]$ (in $\boldsymbol{x}$ if $k=1$ and in $\boldsymbol{y}$ if $k=2$) and perform the knot insertion on $B[\boldsymbol{x},\boldsymbol{y}]$. The resulting generated B-splines would still have support on $\mathcal{M}$ and eventually they would also have minimal support on $\mathcal{M}$. The LR B-splines are generated throughout the construction of an LR-mesh following this procedure.

Definition 5.3. Given a spline LR-mesh $\mathcal{N} = (\mathcal{M},\mu,\boldsymbol{p})$ with $\mathcal{M} = \mathcal{M}_N$ final mesh of a mesh sequence as described in Definition 2.6, the **LR B-spline set** $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ is provided algorithmically as follows. We start by considering the set $\mathcal{B}_1$ of standard B-splines on the initial coarse tensor mesh $\mathcal{M}_1$. Then, for any intermediate step $\mathcal{M}_{i+1} = \mathcal{M}_i + \gamma_i$ with $i = 1,\dots,N-1$ in the construction of the LR–mesh, we produce a new set of B-splines $\mathcal{B}_{i+1}$ by the following algorithm:

1. initialize $\mathcal{B}_{i+1} \leftarrow \mathcal{B}_i$,
2. as long as there exists $B[\boldsymbol{x}^j,\boldsymbol{y}^j] \in \mathcal{B}_{i+1}$ with no minimal support on $\mathcal{M}_{i+1}$,

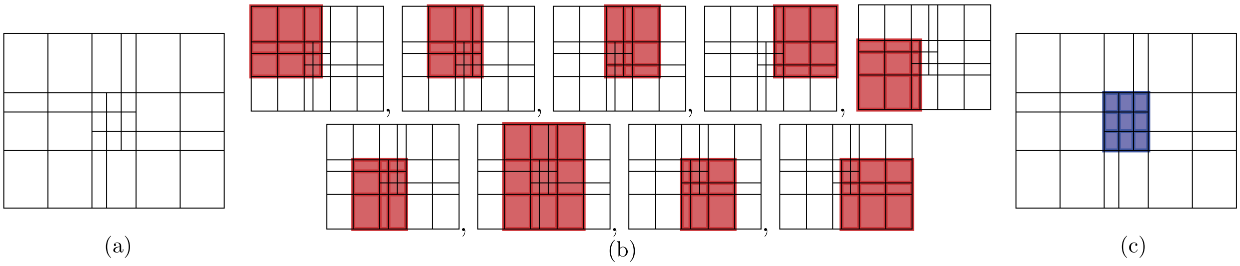

Fig. 4. (a) An LR-mesh $\mathcal{M}$ of multiplicity 1. (b) Supports of the biquadratic LR B-splines defined on $\mathcal{M}$. (c) Support of a minimal support B-spline on the mesh not in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$.

(a) apply knot insertion: $\exists B[\boldsymbol{x}_1^j, \boldsymbol{y}_1^j], B[\boldsymbol{x}_2^j, \boldsymbol{y}_2^j] : B[\boldsymbol{x}^j, \boldsymbol{y}^j] = \alpha_1 B[\boldsymbol{x}_1^j, \boldsymbol{y}_1^j] + \alpha_2 B[\boldsymbol{x}_2^j, \boldsymbol{y}_2^j]$,

(b) update the set: $\mathcal{B}_{i+1} \leftarrow (\mathcal{B}_{i+1} \backslash \{B[\boldsymbol{x}^j, \boldsymbol{y}^j]\}) \cup \{B[\boldsymbol{x}_1^j, \boldsymbol{y}_1^j], B[\boldsymbol{x}_2^j, \boldsymbol{y}_2^j]\}$,

3. $\mathcal{B}^{\mathcal{LR}}(\mathcal{N}) := \mathcal{B}_N$.

Remark 5.4. For any spline LR-mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, $\operatorname{span} \mathcal{B}^{\mathcal{LR}}(\mathcal{N}) \subseteq \operatorname{span} \mathcal{B}^{\mathcal{MS}}(\mathcal{N}) \subseteq \mathbb{S}(\mathcal{N})$. If $\mathcal{M}$ is a tensor mesh then $\mathcal{B}^{\mathcal{LR}}(\mathcal{N}) = \mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ and they are nothing more than the standard bivariate B-splines. The Curry-Schoenberg Theorem ensures that $\operatorname{span} \mathcal{B}^{\mathcal{LR}}(\mathcal{N}) = \operatorname{span} \mathcal{B}^{\mathcal{MS}}(\mathcal{N}) = \mathbb{S}(\mathcal{N})$ and the elements of $\mathcal{B}^{\mathcal{LR}}(\mathcal{M}) = \mathcal{B}^{\mathcal{MS}}(\mathcal{M})$ are linearly independent. However, there are other cases where this equality holds; we will see them in the next section.

After performing the LR B-splines generation algorithm, the functions created will generally not sum to one. For this reason, in Dokken et al. (2013, Section 7) is provided a procedure for positive scaling weights of the LR B-splines to reinstate the partition of unity.

Example 5.5 *($\mathcal{B}^{\mathcal{LR}}(\mathcal{N}) \neq \mathcal{B}^{\mathcal{MS}}(\mathcal{N})$).* In Fig. 4(a) we have an LR-mesh $\mathcal{M}$ of multiplicity 1. Suppose $\boldsymbol{p} = (2, 2)$. This mesh is obtained by inserting two 2-splits and two 1-splits in a tensor mesh $\mathcal{M}_1$. In Fig. 4(b) we see the supports of the LR B-splines on $\mathcal{M}$, i.e., the elements of $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$, with $\mathcal{N} = (\mathcal{M}, 1, (2, 2))$, obtained by refining the B-splines with no minimal support during the insertion of the splits. However if we look at the final mesh $\mathcal{M}$ in Fig. 4(a), we see that there is one MS B-spline, whose support is depicted in Fig. 4(c), not in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$, defined on the mesh.

6. Hand-in-hand principle

In this section we describe the spanning properties of the sets $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ and $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$. Any LR-mesh $\mathcal{M} = \mathcal{M}_N$ is defined through a sequence $\mathcal{M}_{i+1} = \mathcal{M}_i + \gamma_i$ starting from a tensor mesh $\mathcal{M}_1$. We know that on $\mathcal{N}_1 = (\mathcal{M}_1, \mu_1, \boldsymbol{p})$, $\operatorname{span} \mathcal{B}^{\mathcal{MS}}(\mathcal{N}_1) = \mathbb{S}(\mathcal{N}_1)$ as well as $\operatorname{span} \mathcal{B}^{\mathcal{LR}}(\mathcal{N}_1) = \mathbb{S}(\mathcal{N}_1)$. We want to preserve these equalities throughout the construction of $\mathcal{M}_N$ for two reasons. First, we maximize the approximation power of the considered B-splines because the full spline space is spanned, and second, since we have a dimension formula for the spline space, we can use it to determine if the B-splines are linearly dependent or not. Indeed, since they span the whole spline space, if there are more B-splines than the dimension, they must be linearly dependent.

Definition 6.1. Given a spline LR-mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, assume that $\operatorname{span} \mathcal{B}^{\mathcal{MS}}(\mathcal{N}) = \mathbb{S}(\mathcal{N})$, or $\operatorname{span} \mathcal{B}^{\mathcal{LR}}(\mathcal{N}) = \mathbb{S}(\mathcal{N})$ respectively. Let γ be a new split and let $\mathcal{N} + \gamma = (\mathcal{M} + \gamma, \mu + \mu_\gamma, \boldsymbol{p})$ be the refined spline mesh. We say that $\mathcal{N} + \gamma$ goes **MS-wise**, or **LR-wise** respectively, **hand-in-hand** with $\mathcal{N}$ if $\operatorname{span} \mathcal{B}^{\mathcal{MS}}(\mathcal{N} + \gamma) = \mathbb{S}(\mathcal{N} + \gamma)$, or $\operatorname{span} \mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma) = \mathbb{S}(\mathcal{N} + \gamma)$ respectively.

In other words, going hand-in-hand means that if the considered B-splines on the spline mesh $\mathcal{N}$ span the whole spline space $\mathbb{S}(\mathcal{N})$, then also the refined B-splines defined on $\mathcal{N} + \gamma$ will span the refined spline space $\mathbb{S}(\mathcal{N} + \gamma)$.

Remark 6.2. If $\mathcal{N} + \gamma$ goes LR-wise hand-in-hand with $\mathcal{N}$, then it also goes MS-wise hand-in-hand with $\mathcal{N}$. This is trivial because $\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma) \subseteq \mathcal{B}^{\mathcal{MS}}(\mathcal{N} + \gamma)$. The converse is not true in general.

In order to keep spanning the spline space during the construction of an LR-mesh, we have to ensure that all the intermediate spline meshes go MS-wise, or LR-wise, hand-in-hand. A condition to achieve this is stated in the following result, which is a reformulation of Dokken et al. (2013, Theorem 5.10).

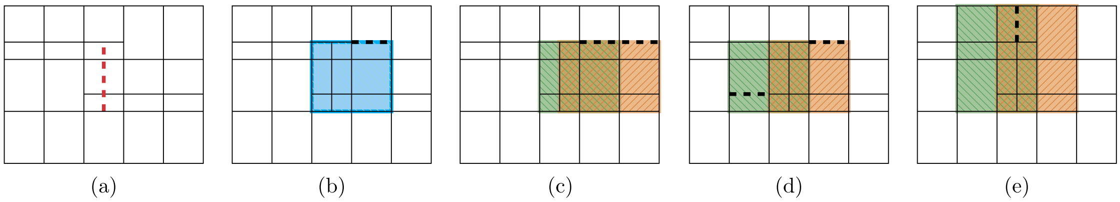

Fig. 5. (a) LR-mesh $\mathcal{M}$ of multiplicity 1 and a new split (dashed) to insert. (b) modification of $\mathcal{M}$ (dashed) to go MS-wise hand-in-hand. (c) and (d) modification of $\mathcal{M}$ (dashed) to go LR-wise hand-in-hand. (e) extension of the new split to go LR-wise hand-in-hand.

Theorem 6.3. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline LR-mesh. Assume that span $\mathcal{B}^{\mathcal{MS}}(\mathcal{N}) = \mathbb{S}(\mathcal{N})$, or span $\mathcal{B}^{\mathcal{LR}}(\mathcal{N}) = \mathbb{S}(\mathcal{N})$ respectively. Let γ be a new k-split to insert and $\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}$ be the expanded spline knot vector on it. Let $\mathcal{B}^{\mathcal{MS}}(\gamma)$, or $\mathcal{B}^{\mathcal{LR}}(\gamma)$ respectively, be the collections of the new B-splines created in the MS, or LR, B-spline set after the insertion of γ. For any $B \in \mathcal{B}^{\mathcal{MS}}(\gamma)$, or $B \in \mathcal{B}^{\mathcal{LR}}(\gamma)$ respectively, let B_γ be the univariate B-spline in the y variable if $k = 1$ or in the x variable if $k = 2$, in the expression of B as in Definition 4.4. Then $\mathcal{N} + \gamma$ goes MS-wise, or LR-wise respectively, hand-in-hand with $\mathcal{N}$ if and only if*

$$\textit{span}\,\{B_\gamma\}_{B\in\mathcal{B}^{\mathcal{MS}}(\gamma)\,(\text{or }\mathcal{B}^{\mathcal{LR}}(\gamma)\ \textit{resp.})} = \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}).$$

Theorem 6.3 allows to check the hand-in-hand of the meshes by looking at the span of univariate B-splines. Note that, since all the B_γ are contained in $\mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}})$, we always have

$$\dim \textit{span}\{B_\gamma\}_{B\in\mathcal{B}^{\mathcal{MS}}(\gamma)\,(\text{or }\mathcal{B}^{\mathcal{LR}}(\gamma)\ \text{resp.})} \le \dim \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}}).$$

We distinguish two cases when this is a strict inequality:

1. The cardinality of $\mathcal{B}^{\mathcal{MS}}(\gamma)$, or $\mathcal{B}^{\mathcal{LR}}(\gamma)$ respectively, is less than $\dim \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}})$,
2. the cardinality of $\mathcal{B}^{\mathcal{MS}}(\gamma)$, or $\mathcal{B}^{\mathcal{LR}}(\gamma)$ respectively, is at least equal to $\dim \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}})$ but the linearly independent univariate B-splines B_γ are less than such dimension.

The cardinality of $\mathcal{B}^{\mathcal{MS}}(\gamma)$, or $\mathcal{B}^{\mathcal{LR}}(\gamma)$, depends on the mutual position of the splits in $\mathcal{M} + \gamma$. However, by slight modifications of the mesh or by extending γ we can always guarantee that $\mathcal{B}^{\mathcal{MS}}(\gamma)$ and $\mathcal{B}^{\mathcal{LR}}(\gamma)$ have at least $\dim \mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}})$ elements, as explained in Fig. 5. There we consider bidegree $\boldsymbol{p} = (2, 2)$ and a 1-split γ to insert into the LR-mesh $\mathcal{M}$ of multiplicity 1 as shown in Fig. 5(a). Since the expanded spline knot vector on γ has length 4, $\dim \mathbb{S}(\mathcal{N} + \gamma) = \dim \mathbb{S}(\mathcal{N}) + 1$ by Theorem 3.9. Therefore, a new B-spline of the considered kind must be generated to have $\mathcal{N} + \gamma$ going MS-wise or LR-wise hand-in-hand with $\mathcal{N}$.

Unfortunately, no B-splines are created after the insertion due to the splits mutual position. Thus $\mathcal{B}^{\mathcal{MS}}(\mathcal{N} + \gamma) = \mathcal{B}^{\mathcal{MS}}(\mathcal{N})$, $\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma) = \mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ and $\mathcal{N} + \gamma$ cannot go neither LR-wise nor MS-wise hand-in-hand with $\mathcal{N}$. However, if we extend by one meshline a split on $\mathcal{N}$, we create a new MS B-spline when inserting γ, whose support is highlighted in Fig. 5(b). In this case, $\mathcal{N} + \gamma$ goes MS-wise hand-in-hand (but not LR-wise). Instead, if we extend by two meshlines the same split, as in Fig. 5(c), or we extend by one meshlines both the splits, as in Fig. 5(d), there is an LR B-spline on the mesh to refine after the insertion of γ and $\mathcal{N} + \gamma$ goes LR-wise hand-in-hand with $\mathcal{N}$. Another strategy is to extend γ. Indeed, if we decide to insert γ one meshline longer, as in Fig. 5(e), then the spline space increases by 2 for Theorem 3.9 but $\mathcal{N} + \gamma$ goes LR-wise, and so MS-wise, hand-in-hand with $\mathcal{N}$ because the two LR B-splines with supports in the upper left and upper right corner of $\mathcal{M}$ will be refined.

However, although the cardinality of such sets is sufficiently large, the linearly independent univariate B-splines B_γ can be insufficient for spanning the whole spline space $\mathbb{S}(\boldsymbol{\tau}_{p_{3-k}}^{\tilde{\mu}_{3-k}})$. An example is reported in Fig. 6. We again consider bidegree $(2, 2)$, an LR-mesh $\mathcal{M}$ of multiplicity 1 and a new 2-split γ as shown in Fig. 6(a).

The expanded spline knot vector on γ has length 7 so that $\dim \mathbb{S}(\mathcal{N} + \gamma) = \dim \mathbb{S}(\mathcal{N}) + 4$ by Theorem 3.9. Moreover, it is easy to check that $\mathcal{N}$ can be constructed LR-wise hand-in-hand. Therefore, $\mathcal{B}^{\mathcal{LR}}(\mathcal{N}) = \mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ and they span the spline space $\mathbb{S}(\mathcal{N})$. When γ is inserted, there are 5 LR B-splines, B^1, B^2, B^3, B^4, B^5, in $\mathcal{B}^{\mathcal{LR}}(\gamma)$, whose support is depicted in Fig. 6(b). The cardinalities $|\mathcal{B}^{\mathcal{LR}}(\gamma)|, |\mathcal{B}^{\mathcal{MS}}(\gamma)|$ are therefore large enough for $\mathcal{N} + \gamma$ to go hand-in-hand with $\mathcal{N}$. However, if we look at the univariate B-splines B_γ, depicted in Fig. 6(c), we can see that the $B_\gamma^4 = B_\gamma^5$ and B_γ^3 can be easily written, via knot insertion, as a linear combination of B_γ^1, B_γ^2. Thus, there are only 3 linearly independent B-splines in $\{B_\gamma\}_{B\in\mathcal{B}^{\mathcal{LR}}(\gamma)}$ and the spline mesh $\mathcal{N} + \gamma$ cannot go neither LR-wise nor MS-wise hand-in-hand with $\mathcal{N}$.

Nevertheless, if the expanded spline knot vector on the k-split γ has length $p_{3-k} + 2$ or $p_{3-k} + 3$, this phenomenon cannot happen. Indeed, if it has length $p_{3-k} + 2$, the spline space on it has dimension one and there exists at least one B_γ. Similarly, if it has length $p_{3-k} + 3$, the spline space on it has dimension 2 and there are at least two different (and so linearly independent) univariate restrictions B_γ.

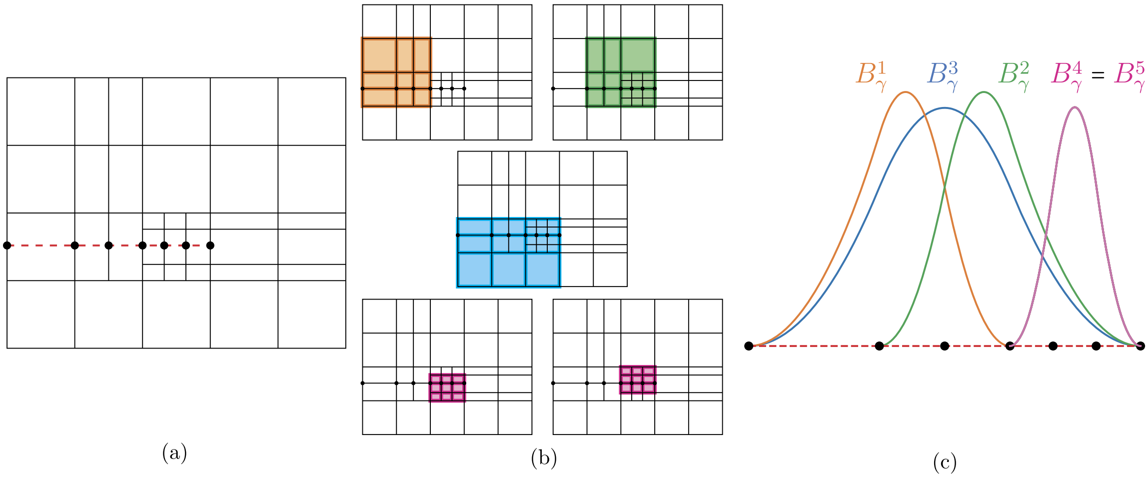

Fig. 6. (a) LR-mesh $\mathcal{M}$ of multiplicity 1 and a new 2-split γ (dashed) with their intersections (black dots). Consider bidegree $(2,2)$. In (b) the supports of the LR B-splines B^1, B^2 (top), B^3 (center), B^4, B^5 (bottom) in $\mathcal{B}^{\mathcal{LR}}(\gamma)$. In (c) their corresponding univariate B-splines.

7. Characterization of linear dependence in $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$

The purpose of this section is to investigate the minimal number of MS B-splines required for a linear dependence relation on a spline mesh $\mathcal{N}$ and features needed in such configurations. In particular, the main results of this section are that at least six MS B-splines are necessary for a linear dependence for any bidegree $\boldsymbol{p} = (p_1, p_2)$ (Proposition 7.15) and that in a configuration of linear dependence with exactly six MS B-splines, one of them is not an LR B-spline (Proposition 7.16). We achieve these results by looking at the minimal number of B-splines needed to satisfy necessary conditions for having a linear dependence relation. First we introduce the *nestedness condition* (Proposition 7.3): at any corner of the region of the mesh where we have linear dependence, there is a B-spline in the linear dependence relation whose support is fully contained in the support of another larger B-spline in the linear dependence relation as well. This implies that the number of B-splines involved in the linear dependence relation is at least five (Corollary 7.4). Then we have to prove that it is impossible to have a linear dependence with only these five. Therefore, first we show the possible arrangements of the supports in case a linear dependence relation has only five B-splines (Lemma 7.5). Then we introduce another necessary condition for linear dependencies regarding the T–vertices in the region of the mesh where the linear dependence occurs (Corollary 7.10). This new condition narrows the possible arrangements of the supports found in Lemma 7.5. Finally, by looking at the position of the five B-splines in this remaining configurations, one can prove Proposition 7.15 mentioned above.

Remark 7.1. We recall that our meaning of linearly dependent functions is slightly different from the standard definition. We consider only functions that are actively linearly dependent, i.e., that have nonzero coefficient in the dependence relation.

Definition 7.2. Given a mesh $\mathcal{M}$ and two MS B-splines $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$, $B[\boldsymbol{x}^2, \boldsymbol{y}^2]$, defined on $\mathcal{M}$, we say that $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$ **is nested into** $B[\boldsymbol{x}^2, \boldsymbol{y}^2]$ if $\operatorname{supp} B[\boldsymbol{x}^1, \boldsymbol{y}^1] \subset \operatorname{supp} B[\boldsymbol{x}^2, \boldsymbol{y}^2]$ and $\operatorname{supp} B[\boldsymbol{x}^1, \boldsymbol{y}^1], \operatorname{supp} B[\boldsymbol{x}^2, \boldsymbol{y}^2]$ share one, and only one, vertex.

Proposition 7.3 *(Nestedness condition). Let $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ be the set of MS B-splines on a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$. Let $\mathcal{B} \subseteq \mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ be a subset of linearly dependent MS B-splines and $\mathcal{R}$ be the region in $\mathbb{R}^2$ given by the union of their supports. Let $(\bar{x}, \bar{y})$ be any (convex) corner in $\mathcal{R}$ and define $\mu_{\bar{x}}$ as the maximal multiplicity that is assigned to $\bar{x}$ among the knot vectors in the x-direction of the B-splines in $\mathcal{B}$. Consider the set*

$$\mathcal{B}_{\mu_{\bar{x}}} := \{B[\boldsymbol{x}, \boldsymbol{y}] \in \mathcal{B} : \bar{x} \in \boldsymbol{x} \text{ with } \mu(\bar{x}) = \mu_{\bar{x}}\}.$$

Define $\mu_{\bar{y}}$ as the maximal multiplicity that is assigned to $\bar{y}$ among the knot vectors in the y-direction of the B-splines in $\mathcal{B}_{\mu_{\bar{x}}}$ and consider the set

$$\mathcal{B}' = \{B[\boldsymbol{x}, \boldsymbol{y}] \in \mathcal{B}_{\mu_{\bar{x}}} : \bar{y} \in \boldsymbol{y} \text{ with } \mu(\bar{y}) = \mu_{\bar{y}}\}.$$

Finally, define $h_x = \min_{B \in \mathcal{B}'} |x_{p_1+2} - x_1|$, $h_y = \min_{B \in \mathcal{B}'} |y_{p_2+2} - y_1|$ and the set of MS B-splines in $\mathcal{B}'$ with smallest support, in both directions:

$$\mathcal{L} = \{B[\boldsymbol{x}, \boldsymbol{y}] \in \mathcal{B}' : |x_{p_1+2} - x_1| = h_x \text{ and } |y_{p_2+2} - y_1| = h_y\}.$$

Then

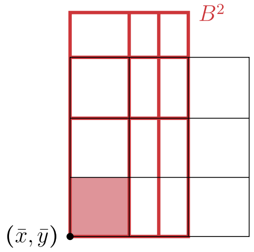

Fig. 7. The support of the two B-splines considered in the proof of Proposition 7.3.

1. *$\mathcal{L}$ has a unique B-spline $B[\boldsymbol{x}^m, \boldsymbol{y}^m]$;*
2. *There exists another B-spline $B[\boldsymbol{x}, \boldsymbol{y}] \in \mathcal{B}'$ such that $B[\boldsymbol{x}^m, \boldsymbol{y}^m]$ is nested into $B[\boldsymbol{x}, \boldsymbol{y}]$.*

Proof. 1. Let us first show that $\mathcal{L} \neq \varnothing$. Consider the element of the box-partition in $\mathcal{R}$ that has $(\bar{x}, \bar{y})$ as vertex. If $\mathcal{L} = \varnothing$, it would mean that such an element in the corner of $\mathcal{R}$ is contained in at least the supports of two B-splines $B^1 = B[\boldsymbol{x}^1, \boldsymbol{y}^1]$, $B^2 = B[\boldsymbol{x}^2, \boldsymbol{y}^2] \in \mathcal{B}'$ such that B^2 is taller than B^1 but narrower as reported in Fig. 7.
Thus, there are $p_2 + 2 - \mu_{\bar{y}}$ horizontal splits of B^1 traversing the interior of $\operatorname{supp} B^2$. Only $p_2 + 1 - \mu_{\bar{y}}$ of them (at most) can be also splits of B^2. This is a contradiction because an extra split traverses the support of B^2 and so it has not minimal support on the mesh. Hence $|\mathcal{L}| \geq 1$. Let us assume there are two MS B-splines in $\mathcal{L}$, $B^1 = B[\boldsymbol{x}^1, \boldsymbol{y}^1]$ and $B^2 = B[\boldsymbol{x}^2, \boldsymbol{y}^2]$. So

$$\begin{aligned} x_1^1 &= x_1^2 & y_1^1 &= y_1^2 \\ x_{p_1+2}^1 &= x_{p_1+2}^2 & y_{p_2+2}^1 &= y_{p_2+2}^2. \end{aligned}$$

If also the internal knots of B^1 and B^2 are the same in both directions, it would mean that $B^2 = B^1$ and there is nothing to prove. Thus, let us assume there is at least one different knot in the x- or y-direction. For instance, suppose there is a different internal knot $x_i^2 \in \boldsymbol{x}^2$ for some i, with respect to $\boldsymbol{x}^1$. Then the corresponding vertical split $\{x_i^2\} \times [y_1^2, y_{p_2+2}^2]$ would traverse the support of B^1. This is a contradiction because B^1 has minimal support.

2. $B^m = B[\boldsymbol{x}^m, \boldsymbol{y}^m]$ is in a linear dependence relation, so the smoothness of it at $\bar{x} \times \mathbb{R}$ and $\mathbb{R} \times \bar{y}$ must be reproduced. Therefore, there must exist at least another B-spline in $\mathcal{B}'$.
Such a MS B-spline $B \in \mathcal{B}'$ cannot be fully contained in the support of B^m because of the minimality of such support. Hence, $\operatorname{supp} B$ exceeds on the right, or on the top, or both on the right and on the top, the support of B^m. By using the same argument adopted to prove that $|\mathcal{L}| \neq \varnothing$, one shows that only the last case can happen. □

Therefore, in every corner of $\mathcal{R}$ there are at least two MS B-splines of the linear dependence relation, one nested into the other. Note that this nestedness condition cannot be satisfied if the mesh considered is an LR-mesh and the bidegree is $(0, 0)$. Indeed, nesting a B-spline into another during the LR–mesh building process would imply to end a split in the middle of an element, which is not allowed. Since Proposition 7.3 is not verified, we conclude that the set of MS (and LR) B-splines of degree $(0, 0)$ is linearly independent on any LR-mesh. On the other hand, it is possible to have nested MS B-splines at the corners of $\mathcal{R}$ in general meshes, even for bidegree (0,0). Fig. 15 (k)–(l), at the end of this section, will illustrate an example of linear dependence for MS B-splines of bidegree (0,0).

Corollary 7.4. *We need at least 5 MS B-splines for a linear dependence relation in $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$.*

Proof. $\mathcal{R}$ has at least four corners and there is a MS B-spline at each of them. The minimal number needed for the nestedness condition is then 5, that is when the 4 MS B-splines at the corners are all nested into the same MS B-spline whose support coincides with $\mathcal{R}$ (see Fig. 8). □

The question now is if five MS B-splines are enough for a linear dependence relation. From the previous results, we know that if so, we have four MS B-splines with supports in the four corners of $\mathcal{R}$ and one larger MS B-spline with support covering the entire region. The rest of this section is devoted to show that five MS B-splines are not enough. For sake of simplicity, we keep the notation used in Fig. 8. So B^1 will be the larger MS B-spline whose support coincides with $\mathcal{R}$ and B^2, B^3, B^4, B^5 are the MS B-splines at the corners ordered clockwise starting from the lower left corner. The knot vectors of B^i, for $i = 1, \ldots, 5$, will be denoted as $\boldsymbol{x}^i = (x_1^i, \ldots, x_{p_1+2}^i)$ and $\boldsymbol{y}^i = (y_1^i, \ldots, y_{p_2+2}^i)$.

In order to have a linear dependence relation, in every point of $\mathcal{R}$ we must have at least two MS B-splines different from zero. In the following Lemma we present how this fact implies spatial relations of the supports in case the linear dependence relation involves only B^1, B^2, B^3, B^4 and B^5.

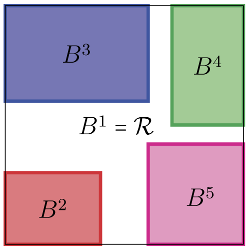

Fig. 8. Configuration with 5 MS B-splines satisfying the nested supports condition for linear dependence.

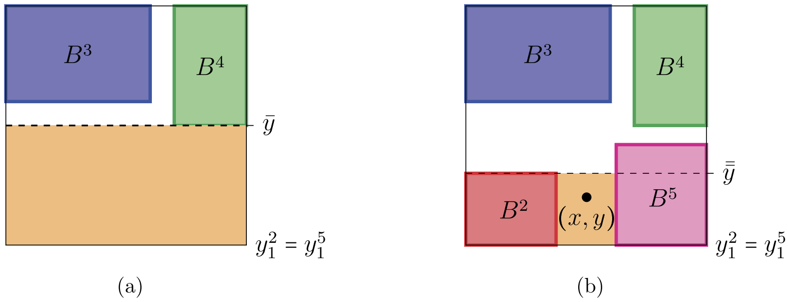

Fig. 9. In (a) the horizontal band of $\mathcal{R}$ not intersected by supp B^3 and supp B^4 is highlighted. In (b), the colored subregion of $\mathcal{R}$ contains a point (x, y) for the proof of the first item of Lemma 7.5.

Lemma 7.5. *Suppose only five MS B-splines are in a linear dependence relation on $\mathcal{R}$. Then*

1. *the supports of B^2 and B^5 intersect each other as well as the supports of B^3 and B^4,*
2. *the supports of B^2 and B^3 intersect each other as well as the supports of B^4 and B^5,*
3. *at least one couple among supp B^2, supp B^4 and supp B^3, supp B^5 intersect each other.*

Proof. Every point in supp B^1 must be inside the support of another B-spline in the linear dependence relation, i.e., the supports of B^2, B^3, B^4, B^5 must be such that there are no white spots left inside $\mathcal{R}$ in Fig. 8.

1. We notice that $y_1^2 = y_1^5$ and, by Proposition 7.3, the y-widths of the supports of B^3 and B^4 must be smaller than the y-width of $\mathcal{R}$, i.e., $y_1^3, y_1^4 > y_1^1$. Let $\bar{y} := \min\{y_1^3, y_1^4\}$. There exists a horizontal band, $[x_1^2, x_{p_1+2}^5] \times (y_1^2, \bar{y}) \subset \mathcal{R}$ that cannot be intersected by supp B^3 and supp B^4 (see the lower band in Fig. 9(a)). We want to prove that supp $B^2 \cap$ supp $B^5 \neq \varnothing$. If supp $B^2 \cap$ supp $B^5 = \varnothing$ then, defined $\bar{\bar{y}} := \min\{\bar{y}, y_{p_2+2}^2, y_{p_2+2}^5\}$, there would exist a point $(x, y) \in (x_{p_1+2}^2, x_1^5) \times (y_1^2, \bar{\bar{y}})$ where none of the four B-splines with supports at the corners of $\mathcal{R}$ would be different from zero (see Fig. 9(b)). (x, y) would only be in the support of B^1. This is a contradiction. An analogous argument yields that supp $B^3 \cap$ supp $B^4 \neq \varnothing$.
2. By exchanging the axes, we can use the same argument adopted in the previous item.
3. Assume the B-splines in the two couples B^2, B^4 and B^3, B^5 do not intersect. Then, since the previous statements are proved, we must have

$$\begin{cases} x_{p_1+2}^3 < x_1^5 \\ y_{p_2+2}^2 < y_1^4 \end{cases} \quad \text{or} \quad \begin{cases} x_{p_1+2}^2 < x_1^4 \\ y_{p_2+2}^5 < y_1^3. \end{cases}$$

   These two cases, depicted in Fig. 10(a)–(b), can be treated in the same way, so we focus only on the first. Consider a point $(x, y) \in (x_{p_1+2}^3, x_1^5) \times (y_{p_2+2}^2, y_1^4)$. Since $x \in (x_{p_1+2}^3, x_1^5)$ we have $(x, y) \notin$ supp B_3, supp B_5. While, since $y \in (y_{p_2+2}^2, y_1^4)$, we have $(x, y) \notin$ supp B^2, supp B^4. Therefore (x, y) is only in supp B^1, which is a contradiction. □

Fig. 11 shows possible arrangements of the supports of B^1, B^2, B^3, B^4, B^5 to satisfy Proposition 7.3 and Lemma 7.5. In Fig. 11(a) the supports of B^3 and B^5 intersect each other while the supports of B^2 and B^4 do not intersect. In Fig. 11(b)–(c) both the pairs at opposing corners of $\mathcal{R}$ intersect each other. In particular, in Fig. 11(c) B^2 is as tall as B^5, B^3 is tall as B^4, B^2 is as wide as B^3 and B^4 is as wide as B^5.

The value of a bivariate B-splines $B[\boldsymbol{x}, \boldsymbol{y}]$ at the lower and left edges of its support can be different from zero if the multiplicity of the knots y_1 and x_1 in $\boldsymbol{y}$ and $\boldsymbol{x}$ is $p_2 + 1$ and $p_1 + 1$ respectively. If one of B_2, B_3, B_4, B_5 is different from zero on an edge of its support then some of the support intersections described in Lemma 7.5 can be just a part of an edge. In particular, this is what happens when $(p_1, p_2) = (0, 0)$. In this case, the intersections described in Lemma 7.5 1.–2.

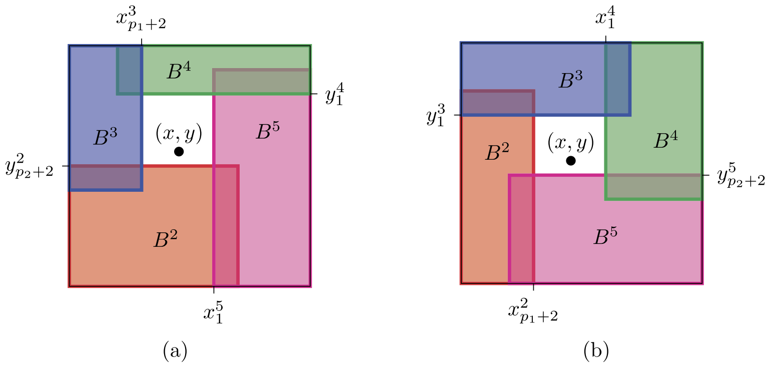

Fig. 10. (a) and (b) are the two possible arrangements of the supports of B^2, B^3, B^4, B^5 inside the support of B^1 when the first two items of Lemma 7.5 hold but not the last.

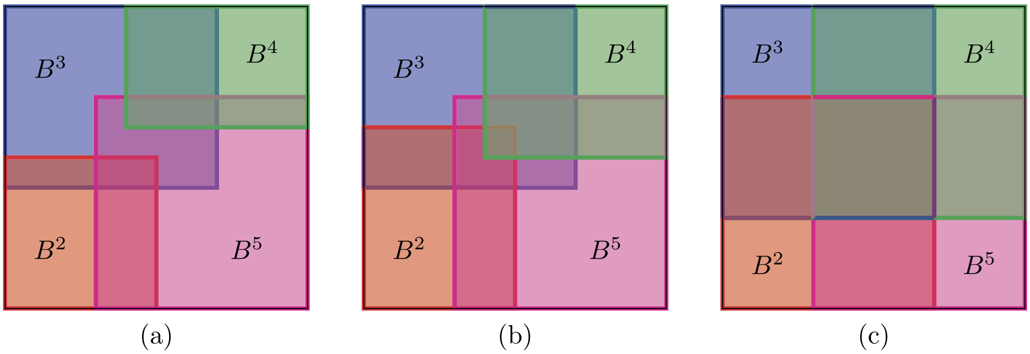

Fig. 11. Arrangements of the four B-splines at corners of suppB^1 satisfying Lemma 7.5.

must be edge intersections in order for the nested B-splines to have minimal support. However, these edge intersections will be aligned, at least in one direction, i.e., there would exist at least one split traversing $\mathcal{R}$ entirely, that is, B^1 would not have minimal support, which is a contradiction. We conclude that 5 MS B-splines are not enough for a linear dependence relation if $(p_1, p_2) = (0, 0)$.

In the rest of this section, for sake of simplicity and briefness, we do not treat the cases with edge intersections. However, the arguments used to get our results can be adapted for these cases by collapsing the regions we consider in our proofs in splits of meshlines of higher multiplicities.

We now investigate more the B-spline support arrangements in the presence of a linear dependence by looking at the T-vertices inside the region $\mathcal{R}$.

Definition 7.6. Let $B[\boldsymbol{x}, \boldsymbol{y}]$ be a MS B-splines on a mesh $\mathcal{M}$. Then its knots define a tensor mesh $\mathcal{M}[\boldsymbol{x}, \boldsymbol{y}]$, as described in Section 4. We define the **meshlines of** $B[\boldsymbol{x}, \boldsymbol{y}]$ as the meshlines in $\mathcal{M}$ forming the tensor mesh $\mathcal{M}[\boldsymbol{x}, \boldsymbol{y}]$ and the **splits of** $B[\boldsymbol{x}, \boldsymbol{y}]$ as the splits in $\mathcal{M}$ made of such meshlines.

Definition 7.7. A vertex $(\bar{x}, \bar{y})$ in $\mathcal{R}$ is called **relevant** if it corresponds to a pair of knots in at least one MS B-spline in the linear dependence relation (see Fig. 12).
A meshline γ is called **relevant** if it is a meshline of a MS B-spline in the linear dependence relation.

An example of relevant vertices and meshlines in a mesh is reported in Fig. 12.

Lemma 7.8. *Any relevant vertex in $\mathcal{R}$ is the intersection of orthogonal relevant meshlines.*

Proof. Let $(\bar{x}, \bar{y})$ be a relevant vertex in $\mathcal{R}$. Then it corresponds to a pair of knots of $B[\boldsymbol{x}, \boldsymbol{y}]$ for some B-spline B involved in the linear dependence. In particular, $(\bar{x}, \bar{y})$ is in the orthogonal splits $[x^j_1, x^j_{p_1+2}] \times \{\bar{y}\}$ and $\{\bar{x}\} \times [y^j_1, y^j_{p_2+2}]$. Therefore there must exist at least 2 orthogonal relevant meshlines contained in such splits intersecting in $(\bar{x}, \bar{y})$. $\square$

Proposition 7.9. *Any relevant meshline is a meshline of at least two MS B-splines in the linear dependence relation.*

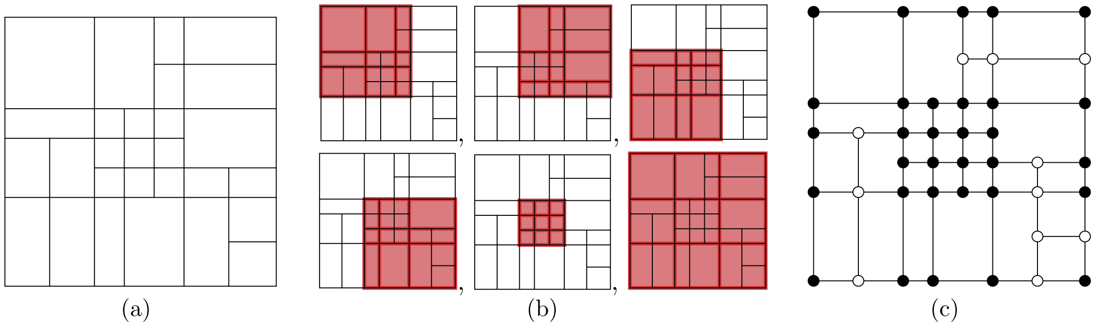

Fig. 12. Consider the mesh on the region $\mathcal{R}$ depicted in (a). Every meshline has multiplicity 1 and consider bidegree (2,2). In (b) we see the supports of the MS B-splines on $\mathcal{R}$. We will prove they are linearly dependent in Example 7.17. In (c) we see the relevant vertices in black and the non-relevant vertices in white.

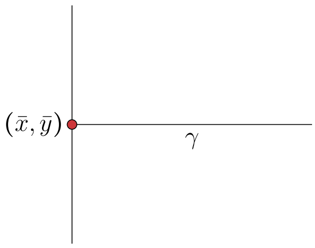

Fig. 13. The T-vertex used in the proof of Corollary 7.10.

Proof. Let $\mathcal{B} = \{B[\boldsymbol{x}^j, \boldsymbol{y}^j]\}_{j=1}^n$ be the set of linearly dependent MS B-splines. Let γ be any k-meshline of $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$. Assume that γ is not a meshline of any other MS B-spline in $\mathcal{B}$. We know that $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$ is $C^{p_k - \mu(\gamma)}$-continuous on γ. The linear dependence relation in $\mathcal{B}$,

$$\alpha_1 B[\boldsymbol{x}^1, \boldsymbol{y}^1](x, y) + \sum_{j=2}^{n} \alpha_j B[\boldsymbol{x}^j, \boldsymbol{y}^j](x, y) = 0 \quad \forall\, (x, y) \in \mathcal{R},$$

can be rewritten expressing $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$ in terms of the others

$$B[\boldsymbol{x}^1, \boldsymbol{y}^1](x, y) = -\frac{1}{\alpha_1} \cdot \sum_{j=2}^{n} \alpha_j B[\boldsymbol{x}^j, \boldsymbol{y}^j](x, y) \quad \forall\, (x, y) \in \mathcal{R} \tag{9}$$

because $\alpha_j \neq 0$ for every $j = 1, \ldots, n$. Consider now $(x, y) \in \gamma \subset \mathcal{R}$. Since γ is not a meshline of any MS B-spline $B[\boldsymbol{x}^j, \boldsymbol{y}^j]$ in $\mathcal{B}$ with $j \geq 2$, the right-hand side is a C^∞-continuous function on γ while the left-hand side is only $C^{p_k - \mu(\gamma)}$-continuous on γ, which is a contradiction. □

Corollary 7.10. *Any relevant T-vertex corresponds to a pair of knots shared by at least two MS B-splines in the linear dependence relation.*

Proof. Let $(\bar{x}, \bar{y})$ be a relevant T-vertex as in Fig. 13. The other three possible cases of T-vertex can be treated similarly.

Since $(\bar{x}, \bar{y})$ is relevant, γ must be relevant from Lemma 7.8. By Proposition 7.9, γ is shared by at least two MS B-splines in the linear dependence relation, $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$, $B[\boldsymbol{x}^2, \boldsymbol{y}^2]$. This means there are two knots $y_r^1 \in \boldsymbol{y}^1$ and $y_s^2 \in \boldsymbol{y}^2$ such that $y_r^1 = \bar{y} = y_s^2$ and $[x_1^1, x_{p_1+2}^1] \times \{\bar{y}\}$, $[x_1^2, x_{p_1+2}^2] \times \{\bar{y}\}$ are splits in the mesh containing γ. Since $(\bar{x}, \bar{y})$ is a T-vertex, it ends such splits, that is, $(x_1^1, y_r^1) = (\bar{x}, \bar{y}) = (x_1^2, y_s^2)$, i.e., $(\bar{x}, \bar{y})$ is a pair of knots shared by $B[\boldsymbol{x}^1, \boldsymbol{y}^1]$ and $B_2[\boldsymbol{x}^2, \boldsymbol{y}^2]$. □

In Section 9 we will see that one can use the previous result to improve the Peeling Algorithm (Dokken et al., 2013, Algorithm 6.3), a tool to check if the LR B-splines considered are linearly independent.

Definition 7.11. Any T-vertex $\boldsymbol{v}$ in a box–partition is the intersection of two colinear meshlines and another meshline γ orthogonal to them. We assign an orientation to these vertices in the following way. We say that the T-vertex $\boldsymbol{v}$ is **downward** if γ is below $\boldsymbol{v}$, **upward** if γ is above $\boldsymbol{v}$, **rightward** if γ is on the right of $\boldsymbol{v}$ and **leftward** if γ is on the left of $\boldsymbol{v}$.

It might happen that a relevant vertex $\boldsymbol{v}$ in $\mathring{\mathcal{R}}$ is a cross-vertex, i.e., the intersection of four meshlines, but one meshline ending in $\boldsymbol{v}$ is not relevant. It means that $\boldsymbol{v}$ behaves as a T-vertex for the B-splines in the linear dependence relation. Therefore, we extend the definition of relevant T-vertex and of its orientation also to these vertices in $\mathring{\mathcal{R}}$.

Theorem 7.12. *Assume five MS B-splines are linearly dependent inside the region $\mathcal{R}$. Then there are at least 4 relevant T-vertices in $\mathring{\mathcal{R}}$, one per orientation.*

Proof. For the sake of simplicity and without loss of generality, we can assume there are only relevant meshlines in $\mathcal{R}$. Referring to any of the examples in Fig. 11, let us consider the vertical splits of B^2 and B^5 in the interior of the support of B^1, i.e., in $\mathring{\mathcal{R}}$. In order to find the minimal number of relevant T-vertices in $\mathring{\mathcal{R}}$, we assume that the parameter values of such vertical splits are the same for B^2 and B^5. We assume the same for B^3 and B^4: the vertical splits of B^4 are included into the vertical splits of B^3.

Suppose first that the multiplicities of the knots in the x-direction corresponding to the vertical edges of $\mathcal{R}$, i.e., x^i_1, for $i = 1, 2, 3$ and $x^i_{p_1+2}$ for $i = 1, 4, 5$, are equal to 1. Then, in $\mathring{\mathcal{R}}$ there are $p_1 + 1$ vertical splits of B^5 and $p_1 + 1$ for B^3, counting the multiplicities. If an end vertex of a vertical split of B^3 or B^5 corresponds to a relevant cross-vertex, it is contained in a split traversing the entire region $\mathcal{R}$, that is, it is contained in a vertical split of B^1. There are p_1 vertical splits of B^1 in $\mathring{\mathcal{R}}$, counting the multiplicities. Therefore, at most p_1 vertical splits in $\mathring{\mathcal{R}}$ of B^3 and B^5 can end with a relevant cross-vertex. Thus there exists at least one relevant vertex of B^5 left on the upper edge of $\operatorname{supp} B^5$ inside $\mathcal{R}$ that cannot be a cross-vertex. The same holds for the relevant vertices in B^3. This proves the existence of two relevant T-vertices in $\mathring{\mathcal{R}}$, one downward and one upward. If the knots in the x-direction corresponding to the vertical edges of $\mathcal{R}$ have higher multiplicities, one can apply the same argument, by subtracting such multiplicities from the count of the vertical splits. Still the difference between the vertical splits in B^3, B^5 and B^1 will be greater than or equal to one and there will be at least one vertical T-vertex per direction necessarily. Applying the same argument to the horizontal splits of B^3 and B^5 we complete the proof. $\square$

Theorem 7.12 holds also if the number of B-splines involved in the linear dependence relation is larger than 5 because of the necessary presence of nested B-splines at the corners.

In order to carry out the proof of the next Proposition 7.15, we need the following definition.

Definition 7.13. Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, let γ be a (k, α)-split in $\mathcal{M}$ for some $k \in \{1, 2\}$. For instance, assume $k = 1$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a spline function in $\mathbb{S}(\mathcal{N})$. F is a piecewise polynomial and therefore, for sufficiently small $\varepsilon > 0$, the functions $F^+ = F_{|(\alpha,\alpha+\varepsilon)\times\mathbb{R}}$ and $F^- = F_{|(\alpha-\varepsilon,\alpha)\times\mathbb{R}}$ are polynomials in x (but splines in y), i.e.,

$$F^+ = \sum_{i=0}^{p_1} f_i^+(y) \cdot (x-\alpha)^i, \qquad F^- = \sum_{i=0}^{p_1} f_i^-(y) \cdot (x-\alpha)^i$$

for f_i^+, f_i^- univariate spline functions. Then we can extend the expression of F^+ and F^- to $\mathbb{R}^2$. We define the **jump function** of F with respect to γ as $J(F)(x, y) = F^+ - F^-$.

Remark 7.14.

- If γ is not in a split traversing the support of F and is not on its boundary, then F is $C^\infty(\gamma)$ and in particular $F^+ = F^-$ so that $J(F)(x, y) = 0$.
- When F is a bivariate B-spline, $F = B[\boldsymbol{x}, \boldsymbol{y}]$ and γ corresponds to a knot in $\boldsymbol{x}$, that is $x_j = \alpha$ for some j and $\gamma = \{x_j\} \times [y_1, y_{p_2+2}]$, then

  $$J(B)(x, y) = J'(B[\boldsymbol{x}])(x) \cdot B[\boldsymbol{y}](y)$$

  where $J'(B[\boldsymbol{x}])(x)$ is a polynomial of the form:

  $$J'(B[\boldsymbol{x}])(x) = \sum_{i=p_1-\mu(\gamma)+1}^{p_1} a_i (x-\alpha)^i.$$

- Let c_1, c_2 be real numbers and F_1, F_2 be spline functions. Then $J(c_1F_1 + c_2F_2)(x, y) = c_1 J(F_1)(x, y) + c_2 J(F_2)(x, y)$.

Proposition 7.15. *We need at least 6 minimal support B-splines for a linear dependence relation in $\mathcal{R}$ for any bidegree.*

Proof. Referring to any configuration in Fig. 11, consider a relevant T-vertex $\boldsymbol{v}$ in B^5. By Corollary 7.10, it has to be shared with at least another MS B-spline. It cannot be shared with B^2 if B^2 is shorter than B^5, and of course it cannot be shared with B^3 or B^4 because it would not be a T-vertex. Then we have two cases:

- There exists a new MS B-spline in the linear dependence relation with support in the y-direction covering the space between the supports of B^5 and B^2 and having $\boldsymbol{v}$ as pair of knots, or
- B^2 is as tall as B^5.

In the first case we have finished the proof. Let us assume then that B^2 is as tall as B^5. Applying the same procedure to the other relevant T-vertices, we either have at least a new MS B-spline in the linear dependence relation, or it must be that B^4 is as tall as B^3, B^2 is as wide as B^3 and B^4 is as wide as B^5. In the first case we have completed the proof. In the second, if no other B-splines are involved, we can write B^1 in terms of B^2, B^3, B^4, B^5:

$$B^1(x,y) = \alpha_2 B^2(x,y) + \alpha_3 B^3(x,y) + \alpha_4 B^4(x,y) + \alpha_5 B^5(x,y) \qquad \text{with } \alpha_j \neq 0. \tag{10}$$

Now, consider any T-vertex downward, corresponding to a 1-split γ in B^2 and B^5. The jump functions of $B^2 = B[\boldsymbol{x}^2, \boldsymbol{y}^2]$ and $B^5[\boldsymbol{x}^5, \boldsymbol{y}^5]$ corresponding to γ, in order to represent B^1 as in equation (10), must satisfy

$$\alpha_2 J'(B[\boldsymbol{x}^2])(x) \cdot B[\boldsymbol{y}^2](y) = -\alpha_5 J'(B[\boldsymbol{x}^5])(x) \cdot B[\boldsymbol{y}^5](y) \tag{11}$$

because B^1 is smooth on γ and there are no other MS B-splines in the linear dependence relation with less regularity in the x-direction on γ. However, the knots of $\boldsymbol{y}^2$ and $\boldsymbol{y}^5$ are different because of the presence of T-vertices leftward and rightward, and equation (11) is impossible to achieve because $B[\boldsymbol{y}^2]$ and $B[\boldsymbol{y}^5]$ are defined on different knots and cannot be proportional everywhere. □

Proposition 7.16. *In a linear dependence relation with six MS B-splines on an LR-mesh, the sixth MS B-spline, B^6, is not an LR B-spline.*

Proof. If B^6 is an LR B-spline it has been obtained through knot insertion from an LR B-spline in a coarser mesh. When the knot insertion is applied the size of the refined B-splines is smaller only in the direction where the knot has been inserted. Therefore, for B^6, in order to be an LR B-spline and be in the linear dependence relation there would exist another B-spline among B^2, B^3, B^4 and B^5 whose support is either as tall or as wide as the support of B^6 and intersects with the support of B^6. Assume we are in the first case of the proof of Proposition 7.15 and there are exactly six MS B-splines in linear dependence. Then there are 4 relevant T-vertices in $\mathring{\mathcal{R}}$ shared with B^6 and identifying the edges of $\operatorname{supp} B^6$. Therefore, $\operatorname{supp} B^6 \subseteq \mathring{\mathcal{R}}$ and cannot be the same as the size of any of B^2, B^3, B^4, B^5 in any direction.

In the second case of the proof of Proposition 7.15, if B^6 is an LR B-spline, we can assume that B^6 is as tall as B^2 and B^5 (the other cases can be treated similarly). Then there would exist a vertical split of B^6 that traverses the support of B^2, or B^5, without being a split of it. This is impossible for the minimality of their supports. □

Example 7.17 *(Linear dependence relation in $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$ with 6 minimal support B-splines).* In this example we prove that 6 MS B-splines are enough for a linear dependence relation for any bidegree $\boldsymbol{p} = (p_1, p_2)$. We start with $\boldsymbol{p} = (2,2)$. Consider the LR-mesh $\mathcal{M}$ of multiplicity one depicted in Fig. 4(a). The supports of the 10 MS B-splines defined on it are represented in Fig. 4(b)–(c). By using the dimension increasing formula in Theorem 3.9, since

- the dimension of the underlying tensor mesh is 3,
- by inserting first the horizontal splits, the expanded spline knot vectors on them have length 4, which results in a dimension increase of 1 per insertion, and
- then, by inserting the two vertical splits, the dimension increases by 2 each time,

we easily compute the dimension of the spline space on $\mathcal{N} = (\mathcal{M}, 1, (2,2))$,

$$\dim \mathbb{S}(\mathcal{N}) = 3 + 1 + 1 + 2 + 2 = 9.$$

Moreover, the construction of $\mathcal{N}$ went LR-wise, and so MS-wise, hand-in-hand. Therefore, we can conclude that there is a linear dependence relation in $\mathcal{B}^{\mathcal{MS}}(\mathcal{N})$. The necessary conditions to be in linear dependence, given in this section, are satisfied by the six MS B-splines whose support is depicted in Fig. 14. Finally, we notice that the 9 LR B-splines on $\mathcal{M}$, reported in Fig. 4(b), are still linearly independent and span the spline space on $\mathcal{N}$.
For any other bidegree $(p_1, p_2) \neq (0,0)$, one can build an LR-mesh preserving the same structure of Fig. 4(a). Fig. 15(a)-(j) shows the cases for $(p_1, p_2) = (3,3), (4,4), (1,1), (1,0), (3,1)$. The insertions are the same as for bidegree (2,2) if $p_k \geq 2$ for some $k \in \{1,2\}$, while if $(p_1, p_2) = (1,0), (0,1), (1,1)$, then it is necessary to use some extensions to get an equivalent arrangement (see the dashed meshlines in the mesh (e) and (g) of Fig. 15). Again the dimension of the spline space is 9 while there are 10 MS B-splines in all the cases. Fig. 15 (k)-(l) shows an equivalent arrangement for bidegree (0,0). However, the mesh in (k) is not an LR-mesh. As we already pointed out, it is not possible to satisfy the necessary nestedness condition for a linear dependence when considering LR-meshes and bidegree (0,0). However, it can be verified on general meshes. For this example, $\dim \mathbb{S}(\mathcal{N}) = 5$ and it is spanned by the characteristic functions of the elements of the box-partition. Therefore, the MS B-splines on the mesh span the spline space but are more than its dimension.

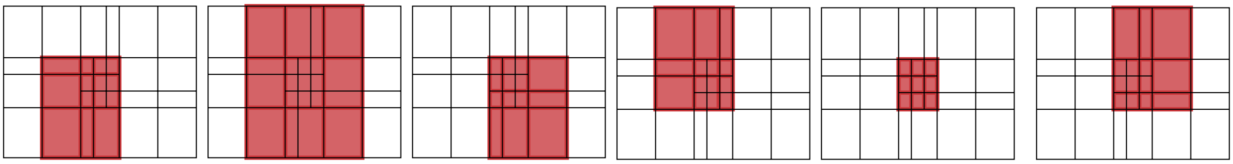

Fig. 14. The supports of the six MS B-splines of degree (2,2) in linear a dependence relation on the LR-mesh depicted in Fig. 4(a).

Fig. 15. In (a) an LR-mesh providing MS B-splines of bidegree (3,3) in an equivalent arrangement of the MS B-splines of bidegree (2,2) on the mesh in Fig. 4(a). In (b) are shown the supports of the six B-splines in the linear dependence relation. In (c)-(d) the same for bidegree (4,4). In (e)-(f) we have the same for bidegree (1,1). Note that we have used two extensions (dashed meshlines) to obtain an equivalent arrangement as for the other bidegrees. In (g)-(h) and (i)-(j) we show the equivalent configuration for bidegrees (1,0) and (3,1). Finally, in (k)-(l) we have a comparable arrangement for bidegree (0,0). However, the mesh in (k) is not an LR-mesh.

8. Minimal number of LR B-splines for a linear dependence relation

In this section we show that at least eight B-splines must be involved for a linear dependence relation in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$. Then we provide examples for any bidegree $\boldsymbol{p} = (p_1, p_2)$ with $p_k \geq 2$ for some $k \in \{1, 2\}$ where the LR B-splines in linear dependence are exactly eight. In such examples the meshes will be refinements of the meshes presented in Example 7.17. As we pointed out in Proposition 7.16, the sixth MS B-spline B^6 in Example 7.17 is not an LR B-spline on the mesh $\mathcal{M}$. In these new examples we show how to refine $\mathcal{M}$ in order to refine B^6 into two B-splines that can now be obtained through the knot insertion algorithm from LR B-splines on coarser meshes. This will move the number of MS B-splines involved in the linear dependence from six to eight but all of them will now be LR B-splines.

Lemma 8.1. *Given a spline LR-mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, assume the elements in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ are linearly independent. If the insertion of a k-split γ causes a linear dependence relation in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)$, then the expanded spline knot vector on γ, $\boldsymbol{\tau}^{\tilde{\mu}_{3-k}}_{p_{3-k}}$, has length at least $p_{3-k} + 3$ and the growth of cardinality is $|\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)| - |\mathcal{B}^{\mathcal{LR}}(\mathcal{N})| > 2$.*

Proof. Theorem 5.2 of Dokken et al. (2013) ensures that if $\boldsymbol{\tau}^{\tilde{\mu}_{3-k}}_{p_{3-k}}$ has length $p_{3-k} + 2$ then the elements in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)$ are linearly independent. Assume that $\boldsymbol{\tau}^{\tilde{\mu}_{3-k}}_{p_{3-k}}$ has length $p_{3-k} + 3$. From the end of Section 6, the refinement goes hand-in-hand only if $|\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)| - |\mathcal{B}^{\mathcal{LR}}(\mathcal{N})| \geq 2$ and there is a linear dependence relation if it is a strict inequality. If the refinement does not go hand-in-hand then it must be $|\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)| - |\mathcal{B}^{\mathcal{LR}}(\mathcal{N})| \leq 1$ and the new B-spline (if existing) is linearly independent of the B-splines in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ as it has a split that intersects γ, which either is not in $\mathcal{M}$ or it has a higher multiplicity in $\mathcal{M} + \gamma$. $\square$

Proposition 8.2. *Given a spline LR-mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$, we need at least 8 LR B-splines for a linear dependence relation in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$.*

Proof. By Proposition 7.3, we must have four nested B-splines at the four corners of $\mathcal{R}$. In order to keep the number of B-splines needed as low as possible we assume as in the proof of Corollary 7.4 that we only need, for the nestedness condition, five B-splines: B^2, B^3, B^4, B^5 contained in B^1. By Theorem 7.12, this implies that there are at least 4 relevant T-vertices. Either

1. the nested B-splines share all these relevant T-vertices between them, or
2. a relevant T-vertex is not shared.

In case 1, we have a configuration as the one reported in Fig. 11(c) and as we have seen in the proof of Proposition 7.16, if there are no more relevant meshlines apart from those of B^2, B^3, B^4, B^5, the other MS B-splines that can be generated in $\mathcal{R}$ using relevant meshlines are not LR B-splines. Therefore, in order to make a linear dependence relation in $\mathcal{R}$, there must exist at least another split that has provided, by Lemma 8.1, a growth in the LR B-spline set of at least three, bringing the number of LR B-splines involved to at least eight. Note that such a split necessarily has refined some of the LR B-splines at the four corners and the LR B-splines generated must all have nonzero coefficients in the linear dependence relation because created via knot insertion from them.

In case 2, there are T-vertices not shared by two B-splines nested at the corners. There must exist other LR B-splines sharing these T-vertices and bringing linear dependence. Hence, there must exist at least another split, aside from those needed for the construction of the nested LR B-splines, that has provided, by Lemma 8.1, a growth of at least three in the LR B-spline set, moving the total number to at least eight. Also in this case, we note that all of these three LR B-splines must have nonzero coefficient in the linear dependence relation for the following reason. One of the three LR B-splines, B, has necessarily a nonzero coefficient because it is used to share a relevant T-vertices. The other two B-splines have been created together with B and are related to it through knot insertion relations. Therefore they also must have nonzero coefficients in the linear dependence relation. $\square$

In the following example we show meshes where there are exactly eight LR B-splines in a linear dependence relation for any bidegree $\boldsymbol{p} = (p_1, p_2)$ with $p_k \geq 2$ for some $k \in \{1, 2\}$. Such meshes are refinements of those presented in Example 7.17.

Example 8.3. Consider the spline mesh $\mathcal{N} = (\mathcal{M}, 1, (2, 2))$ with $\mathcal{M}$ as in Fig. 4(a). We have shown in Example 7.17 that $\dim \mathbb{S}(\mathcal{N}) = 9$ and the construction of $\mathcal{M}$ went LR-wise hand-in-hand. Let us now insert a new split γ, whose expanded spline knot vector has length $p_2 + 3 = 5$, to get the mesh $\mathcal{M} + \gamma$ as shown in Fig. 16(a). Then, by Theorem 3.9, $\dim \mathbb{S}(\mathcal{N} + \gamma) = \dim \mathbb{S}(\mathcal{N}) + 2 = 11$ and $\mathcal{N} + \gamma$ went LR-wise hand-in-hand with $\mathcal{N}$. Furthermore, the LR B-spline set grows by three, $|\mathcal{B}^{\mathcal{LR}}(\mathcal{N} + \gamma)| = |\mathcal{B}^{\mathcal{LR}}(\mathcal{N})| + 3 = 12$ as shown in Fig. 16(b). Therefore, there is a linear dependence relation. The only eight LR B-splines satisfying Proposition 7.3 and Corollary 7.10 are depicted in Fig. 16(c).

For what concerns the general bidegree (p_1, p_2), if $p_k \geq 2$ for some $k \in \{1, 2\}$ it is always possible to arrange the LR B-splines in the same way as for bidegree (2,2). For instance, in Fig. 17 are reported the cases for $(p_1, p_2) = (3,3)$, (4,4), (3,1), (2,0). Also here $\dim \mathbb{S}(\mathcal{N}) = 11$ while $|\mathcal{B}^{\mathcal{LR}}(\mathcal{N})| = 12$. For $(p_1, p_2) = (0,1)$, (1,0) and (1,1) we are unable to

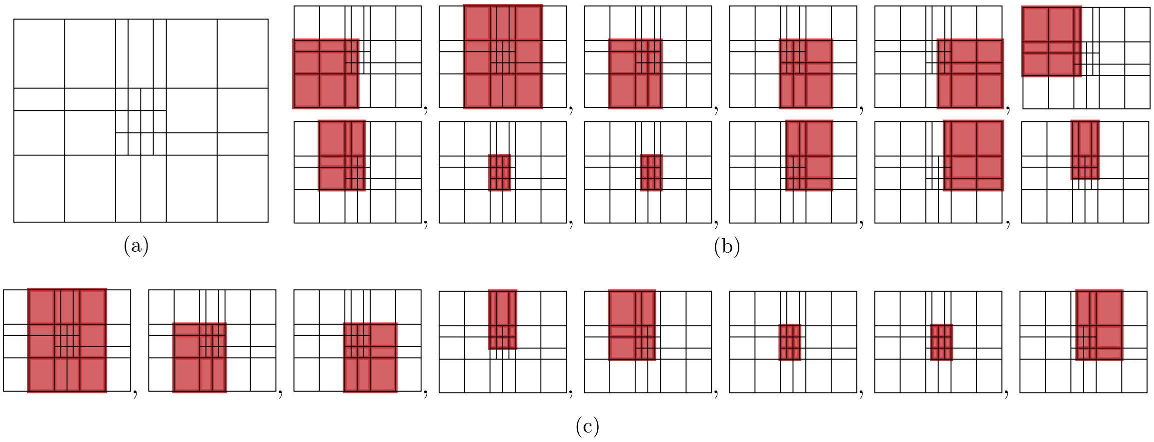

Fig. 16. (a) An LR-mesh of multiplicity 1, (b) the LR B-splines of degree (2,2) on it, (c) the LR B-splines in the linear dependence relation.

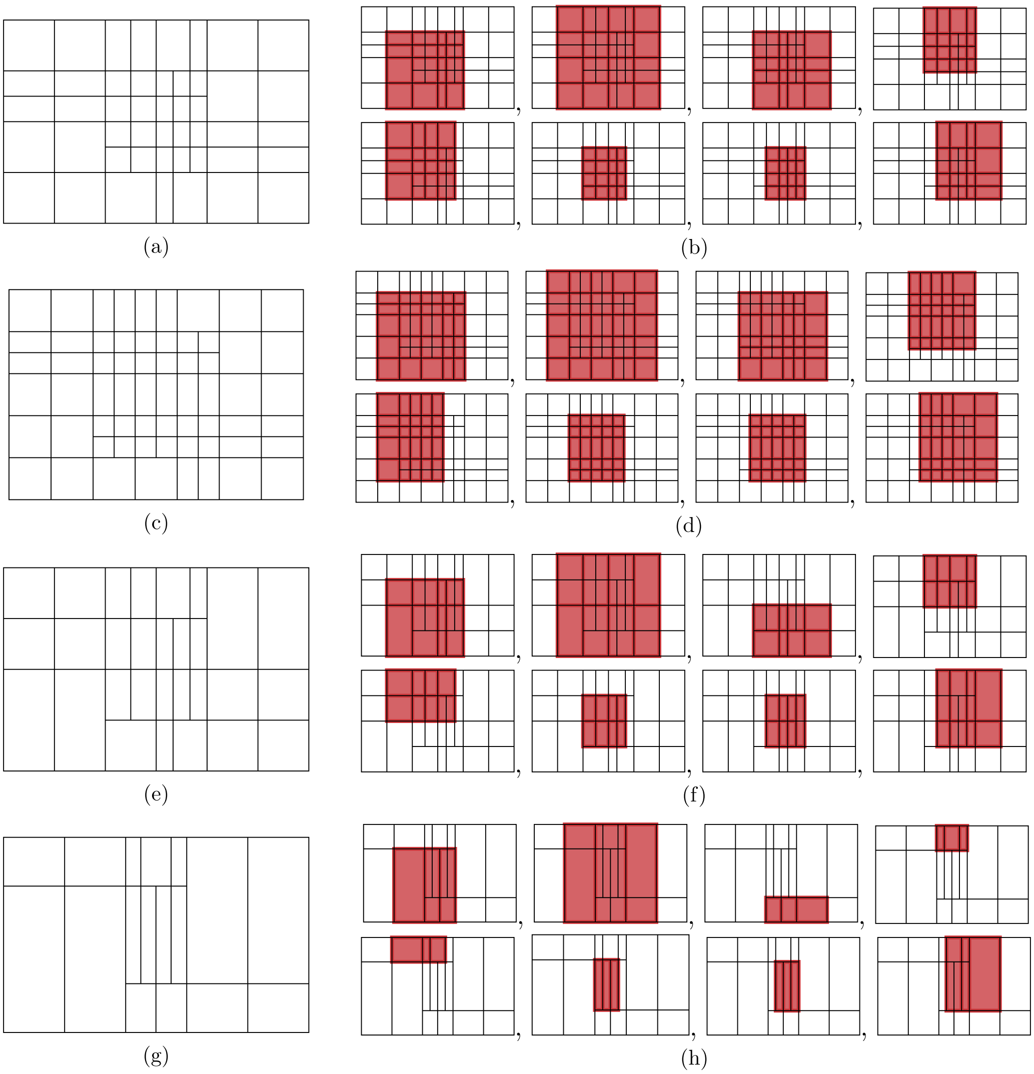

Fig. 17. In (a) is shown an LR-mesh providing LR B-splines of degree (3,3) in an equivalent arrangement of the LR B-splines of bidegree (2,2) on the mesh in Fig. 16 (a). In (b) are shown the supports of the eight B-splines in the linear dependence relation. In (c)-(d) are shown the same for bidegree (4,4), in (e)-(f) for bidegree (3,1) and (g)-(h) for bidegree (2,0).

find an LR B-spline refinement process so that one can insert, via knot insertion, an LR B-spline inside $\mathring{\mathcal{R}}$, to share the relevant T-vertices of the four nested B-splines, without traversing the larger B-spline B^1 with an extra split. This split destroys the linear dependence relation by triggering a refinement of B^1. We conjecture that it is impossible to have a linear dependence relation in $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ for such low bidegrees.

We stress that $\mathcal{M}+\gamma$ in Fig. 16(a) is obtained by refining the mesh $\mathcal{M}$ in Fig. 4(a) considered in Example 7.17. What happens is that with the insertion of a new split, the MS B-spline in the center of mesh $\mathcal{M}$, B^6, is refined into two MS B-splines that can now be obtained through the knot insertion procedure.

9. Improvement of the Peeling Algorithm

The Peeling Algorithm introduced in Dokken et al. (2013) is a tool to check if the LR B-splines on a given LR-mesh are linearly independent. However it does not handle every possible configuration, that is, it might end without answering whether the LR B-spline collection is linearly independent or not. In this section, we briefly recall it and we show how it can be improved, by using Corollary 7.10, to sort out more cases.

Definition 9.1. An element of the box-partition $\mathcal{E}$ is **overloaded** if it is in the support of more B-splines than necessary for spanning the corresponding polynomial space $\Pi_{\boldsymbol{p}}$, that is, it is in more than $(p_1+1)(p_2+1)$ supports. We call a B-spline overloaded if all the elements in its support are overloaded.

An extra B-spline, in a linear dependence, can be removed without changing spanning properties over the elements of $\mathcal{E}$ in its support. So, only overloaded B-splines occur in linear dependencies. A linear dependence relation has to involve at least two overloaded B-splines on every element. Therefore, if on an element there is the support of only one overloaded B-spline, such B-spline cannot be active in a linear dependence. This simple observation is the basis of the Peeling Algorithm.

Peeling Algorithm

1 From the set of LR splines $\mathcal{B}^{\mathcal{LR}}(\mathcal{N})$ create the set $\mathcal{B}^O$ of overloaded B-splines;
2 Let $\mathcal{E}^O$ be the elements of $\mathcal{E}$ in the supports of the B-splines in $\mathcal{B}^O$;
3 Initialization of a subset $\mathcal{B}_1^O$ of $\mathcal{B}^O$ we are going to define, $\mathcal{B}_1^O=\varnothing$;
4 **for** *every element β in $\mathcal{E}^O$* **do**
5   **if** *only one B-spline B of $\mathcal{B}^O$ has β in its support* **then**
6         $\mathcal{B}_1^O=\mathcal{B}_1^O\cup\{B\}$
7 **if** $\mathcal{B}^O\backslash\mathcal{B}_1^O=\varnothing$ **then**
8   linear independence.
9 **else**
10   **if** $\mathcal{B}_1^O=\varnothing$ **then**
11   break, but might have linear dependence.
12     $\mathcal{B}^O=\mathcal{B}^O\backslash\mathcal{B}_1^O$;
13   Go to 2;

The implementation of it is described in Dokken et al. (2013) in terms of matrices.

However, it might happen that every element of $\mathcal{E}^O$ is shared but yet the overloaded LR B-splines are linearly independent. An example is reported in Fig. 18. We consider bidegree (2,2) and an LR-mesh of multiplicity one. In the highlighted region in Fig. 18(a) there are the supports of five LR B-splines, reported in Fig. 18(b), that form the collection $\mathcal{B}^O$ of the algorithm. Then, for each element of the box-partition in such region we count how many of these supports are on it. If an element is only in one support, the corresponding B-spline is placed in the subcollection $\mathcal{B}_1^O$ of the algorithm. From Fig. 18(c), we see that $\mathcal{B}_1^O=\varnothing$. Therefore, the algorithm stops without answering whether the LR B-splines on the mesh are linearly independent or not. However, if we now look at the T-vertices in the region, highlighted in Fig. 18(c), we see that none of them is shared, as pair of knots, in two or more B-splines of $\mathcal{B}^O$. Since the necessary condition for linear dependency Corollary 7.10 is not satisfied, we can conclude that the LR B-splines on the mesh are linearly independent.

The Peeling Algorithm can therefore be improved by inserting in $\mathcal{B}_1^O$ also the B-splines of $\mathcal{B}^O$ that have an exclusive T-vertex as pair of knots. Furthermore, if the cardinality of $\mathcal{B}^O$ becomes less than 8 at any iteration of the algorithm, we can conclude that the LR B-spline collection is linearly independent thanks to Proposition 8.2.

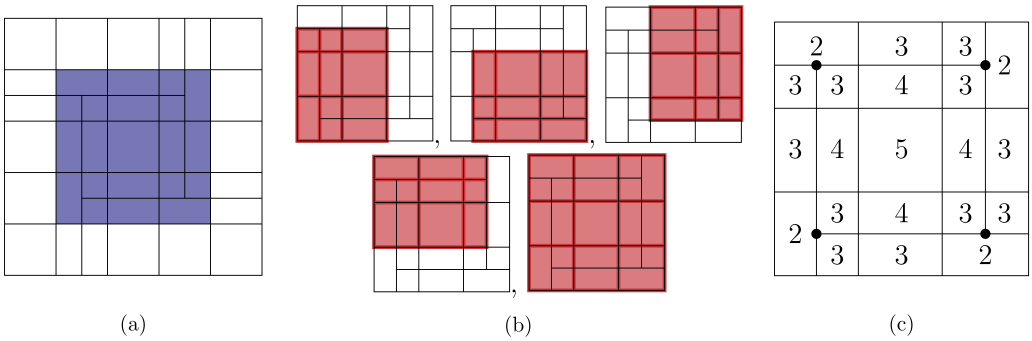

Fig. 18. Consider bidegree (2,2). In the highlighted region in (a) there are the supports of five overloaded LR B-splines, depicted in (b). The numbers in the elements of the region, reported in (c), indicate how many supports of these B-splines are on them. The highlighted vertices are the T-vertices corresponding to pair of knots of the overloaded LR B-splines.

10. Conclusions, conjectures and future work

In this work we have identified necessary features of the mesh to have a linear dependence relation in the MS and LR B-spline sets for any bidegree $\boldsymbol{p}$. Namely, if the union of the supports of the B-splines involved in the linear dependence relation is denoted as $\mathcal{R}$,

- there are nested B-splines at the corners of $\mathcal{R}$, and
- every relevant T-vertex is shared.

Moreover, we have proved that the minimal number of MS B-splines needed for a linear dependence relation is six while for the LR B-splines is eight. These numbers are sharp for any bidegree $\boldsymbol{p} = (p_1, p_2)$ for the MS B-splines and for the LR B-splines with $p_k \geq 2$ for some $k \in \{1, 2\}$. When $(p_1, p_2) = (0,1), (1,0)$ or $(1,1)$, we conjecture it is not possible to have a linear dependence relation in the LR B-spline set.

In our future work, we would like to classify the meshes with a linear dependence relation involving this minimal number of MS B-splines. The number of possible cases would then be dependent on the bidegree chosen. Our conjecture is that every possible configuration of linear dependency is a refinement of one of such cases. Note that this is what happens in the meshes of Example 8.3.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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