

PAPER • OPEN ACCESS

A comparison of linear interpolation and spline interpolation for turbine efficiency curves in short-term hydropower scheduling problems

To cite this article: H I Skjelbred and J Kong 2019 *IOP Conf. Ser.: Earth Environ. Sci.* **240** 042011

View the [article online](#) for updates and enhancements.



IOP | ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

A comparison of linear interpolation and spline interpolation for turbine efficiency curves in short-term hydropower scheduling problems

H I Skjelbred and J Kong

SINTEF Energy Research, Sem Sælands Vei 11, 7034 Trondheim, Norway

E-mail: hansivar.skjelbred@sintef.no

Abstract. The purpose of short-term hydropower scheduling (STHS) in a competitive electricity market is to maximize the profit in the near future while adapting to the mid-term strategies for system operation. To ensure a realistic and feasible operation for STHS, it is crucial to accurately represent the hydropower production function (HPF). The turbine efficiency can be represented by a two-dimensional table provided by the turbine manufacturers or measured by the hydropower producers on-site with some given efficiency values for a set of net water head and discharge values. In literature, the actual efficiency value for any working point is usually linearly interpolated from the given efficiency values. To the best of authors' knowledge, no study has been done so far to investigate how a nonlinear interpolation for the turbine efficiency, i.e. spline interpolation, may contribute to the application of STHS. In this paper, we compare the performance of linear interpolation and spline interpolation for turbine efficiency curves in the STHS problem and the bidding strategy in intraday market. The interpolation methods are tested on an operational hydro-power scheduling model, Short-term Hydro Optimization Program (SHOP), which is used by most large hydropower producers in the Nordic countries. Numerical tests demonstrate superior behavior of spline interpolation for the estimation of turbine efficiency between measured points. Compared to linear interpolation, spline interpolation for turbine efficiency curves gives smoother discharge transitions in day-ahead scheduling and is vital for obtaining continuous marginal cost curves in intraday trading. Based on the results in this paper, spline interpolation is a promising alternative to linear interpolation for turbine efficiency curves in STHS.

1. Introduction

In short-term hydropower scheduling (STHS) it is essential to precisely represent the hydropower production function (HPF). The relationship between discharge and electric power output from a generating unit is a nonlinear function. It depends on both the net head and the turbine/generator efficiency. In addition, the turbine efficiency is a nonlinear function of the net head and the water discharge. It can be expressed by a two-dimensional table, known as the Hill chart [1], provided by the turbine manufacturers or measured by the hydropower producers on-site.

In this paper, we focus on the contribution of the accurate representation of turbine efficiency to the STHS problem and to the bidding strategy in intraday market. Special attention is paid to the impact of the different interpolation methods applied to obtain the actual turbine efficiency values. In literature, the actual efficiency values for any working point are usually linearly interpolated from the given efficiency values [1-5]. To the best of authors' knowledge, no study has been done so far to investigate



how a nonlinear interpolation for the turbine efficiency, i.e. spline interpolation, may contribute to the application of STHS.

Spline interpolation is a form of interpolation where the interpolant is a special type of piecewise polynomial called a spline. It can potentially avoid challenges of simple polynomials, like Runge's phenomenon [6]. Spline interpolation can reproduce the value at all points used for the curve fitting, but still have a smoothly changing gradient.

A summary of the steps involved in the process from obtaining to transforming the turbine efficiency values into the STHS problem is presented in Figure 1. In Step 1, the turbine efficiency is normally measured at different levels of discharge in the plant. These measurements can be done on a scaled prototype or on-site. Since data acquisition is much simpler on a prototype model, improved and new methods for scaling efficiency measurements are under development [7, 8]. Both scaled and on-site measurements will result in a set of discrete points, while the hydropower producers need a continuous curve for the full working area of the unit. Curve fitting is hence needed to transform the set of discrete points to a set of continuous curves in Step 2. As described in chapter 3.10.2 of [9], the choice of interpolation methods plays an important role in the construction of turbine efficiency curves. The interpolation for the turbine efficiency curve takes place when the optimization problem of STHS is mathematically formulated in Step 3.

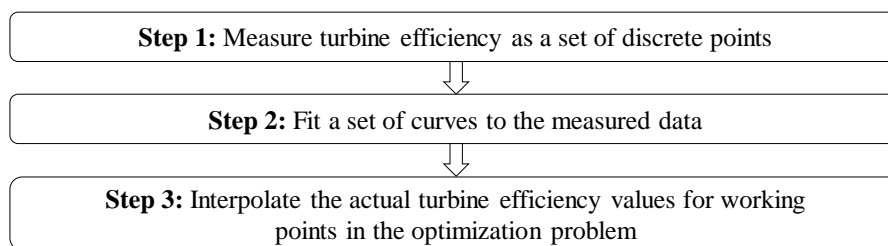


Figure 1. Steps in the process of obtaining and transforming turbine efficiency curves in STHS.

The rest of the paper is organized as follows. In Section 2 we point out the critical issues in the process where the choice of interpolation methods will count. In Section 3 we describe the methods we use to construct the turbine efficiency curve and interpolate the efficiency value for any given point. Section 4 uses a real-world example to show the impact of the different interpolation methods. All the tests are run by an operational STHS tool. Conclusions and suggestions for further work are given in the final section.

2. Problem description

In this section, we discuss how curve fitting and interpolation methods will influence the results of the STHS problem and the determination of the bidding curve for intraday re-planning.

2.1. Turbine efficiency measurements and curve fitting

The challenges related to curve fitting depend on the properties of the underlying data, as explained in the following two sections.

2.1.1. Measurements on a scaled prototype. Due to the relatively low cost and high accessibility, a large number of measurements can be done on a scaled prototype. These measured discrete values do normally not result in a perfect continuous curve, since there will be some noise and uncertainty in the sensors and setup. The process of fitting a curve to these data is a trade-off between smoothing the noise and preserving local characteristics. Annex H in [9] proposes a curve fitting method to balance these two requirements based on splines. The fundamental physical processes affecting the turbine efficiency do not directly favour splines above other methods. The motivation for using splines in this situation is the freedom it gives to adjust the number of piecewise polynomials based on measurement

quality. If there is little noise in the measurements, a few points are enough to define one of the polynomials. If there is much noise, it may take a lot of points to define the it. The result is always a smooth curve. [9] also uses an example to illustrate how this method can be applied in practice.

2.1.2. Measurements on a real plant. Most hydropower plants need their operation to be stopped and have measuring equipment installed to find the turbine efficiency. The down-time and man labour involved is costly. Consequently, on-site measurements of turbine efficiency will not be done very often. In contrast to scaled measurements, only a limited number of points can be measured on-site. Hydropower producers hence need to choose a curve fitting method that retrieves the missing information between the measured points as realistically as possible. The method can be verified by comparing the interpolated values from the retrieved curves with measurements that were not considered when the curve was constructed ("out-of-sample"-testing). In Section 4.2 we present such a test for a real data set.

2.2. Interpolation for turbine efficiency curves

The challenge in Step 3 in Figure 1 is to express the turbine efficiency curve so it fits into the mathematical framework of the STHS tool. The activities in Step 3 do not involve any reduction of noise or estimation of unknown data points. Instead, the main goal is to interpolate the actual turbine efficiency values from the measured points and apply them to formulate the HPF. The accurate representation of HPF will help to bring about precise and optimal results.

2.2.1. Optimal scheduling in the day-ahead market. Since bidding and clearing of the day-ahead market is done several hours before the time of operation, there is enough time for hydropower producers to run a full optimization on a detailed model of their watercourses. In this paper we use the operational scheduling tool SHOP [10] to test the impact of different interpolation methods for turbine efficiency curves. SHOP is a mixed integer linear programming (MILP) model. The HPF is approximated as a convex piecewise linear function in the optimization problem. Splitting the nonlinear function into a number of linearized segments allows for a precise description of turbine efficiency at each breakpoint. These breakpoints are usually not the same as those measured points of turbine efficiency, and therefore, interpolation is needed to obtain the corresponding efficiency values. Linear interpolation of turbine efficiency results in stepwise gradients. By contrast, spline interpolation leads to the continuously changing gradient.

2.2.2. Bidding strategy in intraday re-planning. Bidding on intraday markets is done much closer to real time, so there is rarely enough time to re-optimize the entire hydraulic system. In [11] a method is proposed to build bidding curves for intraday markets. This approach is more computationally efficient than optimization and requires a finer representation of the HPF. In this context the smoothness and continuous gradient of spline interpolation for turbine efficiency curves becomes even more important than in the day-ahead scheduling.

3. Methodology

In this section, we use the mathematical formulation to highlight the impact of interpolation methods for turbine efficiency curves on the STHS problem and bidding strategy. A major simplification is that we disregard all head effects. This means that the plant is assumed to operate with constant head, without head losses and without head-dependency in the turbine efficiency curve. We also set the generator efficiency constant to 100%. Only one unit in the plant is considered. These conditions allow us to focus on the interpolation methods for the turbine efficiency curve in one dimension. The proposed interpolation methods are currently implemented in SHOP without the above-mentioned simplifications. It is in operational use by most large hydropower producers in the Nordic countries.

3.1. Spline interpolation

Firstly, we assume that a set of measured points (also known as "knots") are already available. (Q_n, η_n) refers to the measured water discharge and the corresponding efficiency of the unit at point n (m^3/s , %).

In this paper, the term "spline" means "natural cubic spline". A spline consists of a set of N 3rd order polynomials $f_n(q)$. Each polynomial is fitted to a segment between each pair of the measured points. In other word, the approach of using spline interpolation to mathematically model the turbine efficiency curve fixed by $N + 1$ knots $\{(Q_n, \eta_n): n = 1, 2, \dots, N + 1\}$ is to interpolate between all the pairs of knots (Q_n, η_n) and (Q_{n+1}, η_{n+1}) with polynomials $f_n(q), n = 1, 2, \dots, N$. Figure 2 explains the relationship between knots (Q_n, η_n) and polynomials $f_n(q)$.

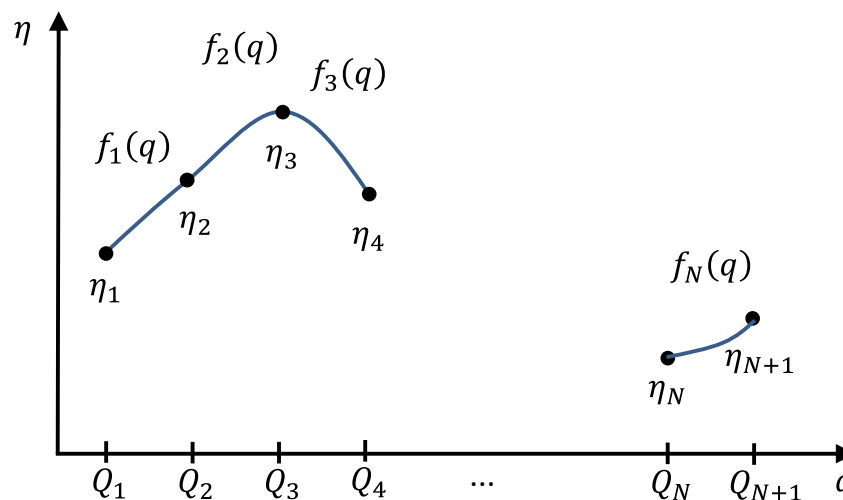


Figure 2. Relationship between knots and polynomials by using spline interpolation.

As the spline will take a shape that minimizes the bending under the constraint of passing through all knots, both first and second derivatives of the polynomials will be continuous everywhere and at the knots. To satisfy this, for knots $n = 2 \dots N$, the following conditions must be met

$$f_n(Q_n) = \eta_n \quad (1)$$

$$f_{n-1}(Q_n) = f_n(Q_n) \quad (2)$$

$$f'_{n-1}(Q_n) = f'_n(Q_n) \quad (3)$$

$$f''_{n-1}(Q_n) = f''_n(Q_n) \quad (4)$$

At the first point of the first polynomial, and the end point of the last polynomial, $f''_1(Q_1)$ and $f''_N(Q_{N+1})$ are set to 0 as a natural boundary condition. That is, the boundary conditions for the first polynomial are defined by

$$f_1(Q_1) = \eta_1 \quad (5)$$

$$f''_1(Q_1) = 0 \quad (6)$$

And the boundary conditions for the last polynomial are formulated as

$$f_N(Q_{N+1}) = \eta_{N+1} \quad (7)$$

$$f''_N(Q_{N+1}) = 0 \quad (8)$$

Based on equations (1) – (8), we can find the polynomial coefficients for the spline. Since each 3rd order polynomial has four unknown coefficients, this results in a set of $4N$ linear equations with $4N$ unknowns. The solution is found by a tridiagonal algorithm taken from chapter 3.3 in [12]. In our implementation we calculate the coefficients for each turbine efficiency spline only once. By storing the coefficients as part of the curve description, the actual efficiency at discharge q can be found as $\hat{\eta}(q) = f(q)$ by a simple and computationally efficient calculation.

3.2. Linear interpolation

Linear interpolation of a set of measured points is rather straightforward. It is defined as the concatenation of linear interpolants between each pair of the measured points. This results in a continuous curve with discontinuous derivatives. The actual efficiency at discharge q that lies between two points (Q_n, η_n) and (Q_{n+1}, η_{n+1}) , $n = 1, 2, \dots, N$ is obtained by

$$\hat{\eta}(q) = \frac{\eta_n Q_{n+1} - \eta_{n+1} Q_n}{Q_{n+1} - Q_n} + \frac{\eta_{n+1} - \eta_n}{Q_{n+1} - Q_n} q \quad (9)$$

3.3. Including turbine efficiency in the MILP framework of the STHS problem

The HPF includes the turbine efficiency into the formulation of the optimization problem. Equation (10) is a version of the HPF with the simplifications mentioned at the beginning of this section.

$$\varphi(q) = G \cdot H \cdot \hat{\eta}(q) \cdot q \quad (10)$$

where

$\varphi(q)$ Power output as a function of the discharge of the unit (MW).

G Conversion constant taking into account the gravity acceleration, water density and makes the appropriate unit conversions from (m) and (m³/s) to (MW), default value is $9.81 \cdot 10^{-3}$.

H Net head of the unit, assumed to be constant in this paper (m).

For robustness and computational efficiency, MILP is chosen as the basis for the optimization model of SHOP. Since q is unknown as a variable in the optimization problem, the interpolated value for $\hat{\eta}(q)$ may in general turn equation (10) into a nonlinear and nonconvex function. In order to incorporate it into the MILP framework, nonconvex areas are replaced with convex ones while nonlinearities are handled by approximating the HPF as a piecewise linear function. The convexification and approximation are out of the scope of this paper. Interested readers are referred to [10] for details on how the HPF is built in SHOP without the simplifications in this paper.

3.4. Including turbine efficiency in intraday bidding

The method used to calculate marginal cost curves for bidding in the intraday market [11] is designed to capture as many details from the turbine efficiency curve as possible. A high level of details is a necessity for building precise bidding curves. The goal is to determine the optimal discharge level for the plant for every possible price in the market.

In economics, marginal cost is the change in the opportunity cost that arises when the quantity produced has an increment of one unit. Therefore, the marginal cost is proportional to the reciprocal of the first derivative of $\varphi(q)$, including $\hat{\eta}(q)$ at the same time. In the actual implementation, we assume that the marginal cost for one operating point is the change in the opportunity cost of water involved as a result of an infinitesimally small increase in the discharge of the unit, typically 0.001 m³/s.

Then the relative change in the marginal cost curve mainly depends on the both first and second derivatives of $\hat{\eta}(q)$. If $\hat{\eta}(q)$ is obtained by spline interpolation where the first and second derivatives of $\hat{\eta}(q)$ are both continuous, the marginal cost curve will also change smoothly. If $\hat{\eta}(q)$ is obtained by linear interpolation where the first derivative at the knots are discontinuous, the marginal cost curve will have a stepwise behaviour, i.e. a jump at each measured point. As a bidding curve used in the

intraday market, a continuously smooth curve is more acceptable, and therefore, spline interpolation for turbine efficiency curves should be applied.

4. Numerical results

In this section, we use a real-world example to illustrate the impact of the different interpolation methods on the results of STHS problem and bidding strategy. All the tests in this paper are based on the efficiency values taken from on-site measurements of an operational Francis turbine. The values from these measurements are listed in Table 1. The tests are run by SHOP on an Intel Core i7-6600U processor with 16 GB of RAM. CPLEX 12.6.3 is the solver.

Table 1. Test data based on actual measurements from an operational Francis turbine.

Point #	Discharge (m ³ /s)	Turbine efficiency (%)
1	12.98	85.29
2	17.38	90.38
3	20.92	93.51
4	24.06	94.42
5	25.42	94.57
6	27.74	94.30
7	29.50	93.41
8	31.06	92.54

4.1. Comparison of turbine efficiency curves built by different interpolation methods

Using the data listed in Table 1 as input, three different methods are applied to create continuous turbine efficiency curves:

M1: A natural cubic spline found by solving equations (1) – (8) for the measured points

M2: A piecewise linear curve defined by straight segments between the measured points

M3: A 4th order polynomial fitted by least square estimation to the measured points

Polynomial interpolation is added as a reference. Figure 3 compares the resulting continuous efficiency curves built by the methods.

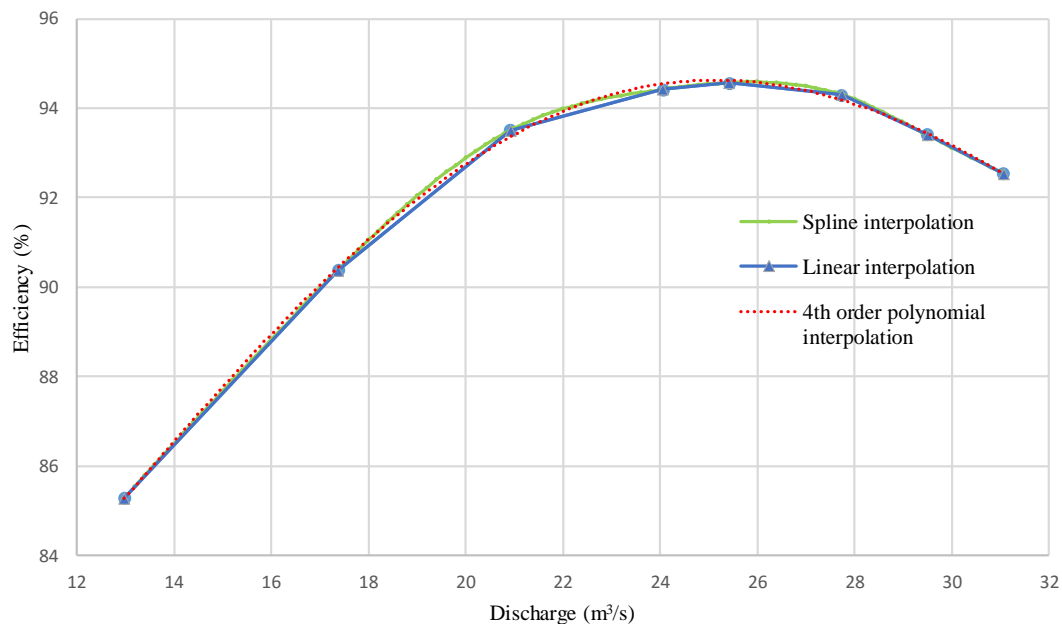


Figure 3. Turbine efficiency curves built by three different interpolation methods.

If we zoom in around the best efficiency point (25.42 m³/s, 94.57%) as in Figure 4, we see more clearly the differences between the three methods. While linear interpolation keeps the best efficiency at the same level of discharge (25.42 m³/s) as in Table 1, it shifts downwards to 25.13 m³/s with polynomial interpolation and upwards to 25.9 m³/s with spline interpolation. We also note that the continuous curves given by linear and spline interpolation pass exactly through each measured point.

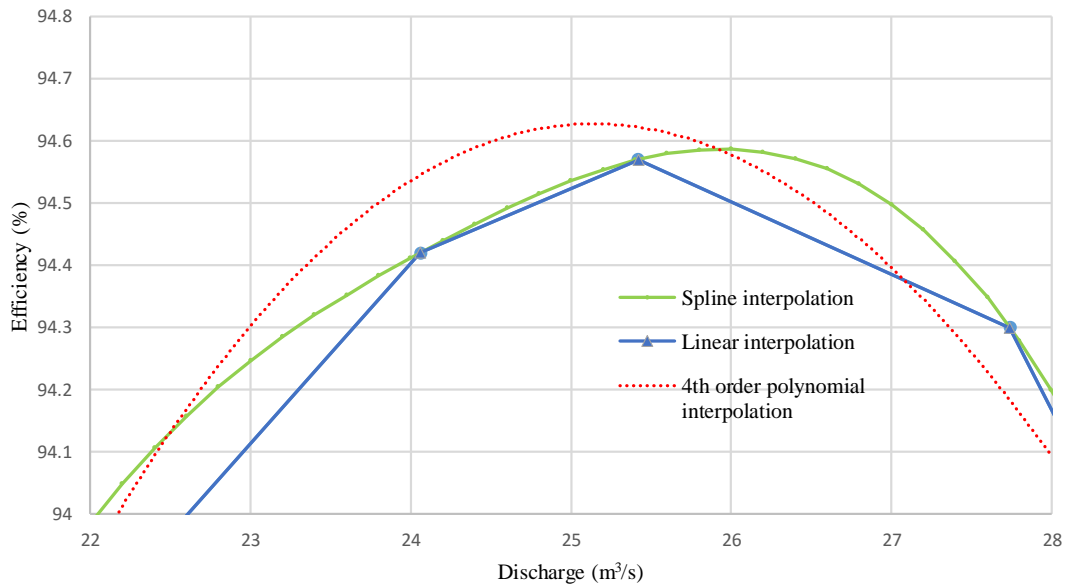


Figure 4. Turbine efficiency curves around best point built by three different interpolation methods.

4.2. Prediction of turbine efficiency between measured points

As mentioned in Section 2.1.2 we can use out-of-sample values to test the prediction quality of the interpolation methods. We want to see how well different methods can reproduce the measured efficiency if one of the points is unknown when the curve is determined.

To test this, we remove points 2–7 one by one from the data set. Then we re-calculate the continuous efficiency curve from the remaining points in the data set by each of the methods listed in Section 4.1. Based on the continuous and fitted curve we interpolate the efficiency for the removed point. Polynomial interpolation is still used as a reference. Both 3rd and 4th order polynomials are fitted to the remaining points using least squares curve fitting. The resulting efficiency for all removed points is listed in the table below.

Table 2. Prediction precision of interpolation methods based on out-of-sample testing. Pol3=3rd order polynomial, Pol4=4th order polynomial.

Removed point #	True	Turbine efficiency (%)				Error (%)			
		Spline	Linear	Pol3	Pol4	Spline	Linear	Pol3	Pol4
2	90.38	90.64	89.84	90.83	91.00	0.26	0.54	0.45	0.62
3	93.51	93.20	92.52	93.15	93.20	0.31	0.99	0.36	0.31
4	94.42	94.50	94.24	94.50	94.63	0.08	0.18	0.08	0.21
5	94.57	94.57	94.37	94.63	94.65	0.00	0.20	0.06	0.08
6	94.30	94.18	93.91	94.25	94.07	0.12	0.39	0.05	0.23
7	93.41	93.50	93.36	93.41	93.46	0.09	0.05	0.00	0.05
Average						0.14	0.39	0.17	0.25

The turbine efficiency found from the interpolated curve is compared to the true efficiency. The difference between the approximated and true efficiency is used to calculate the error. We use the error

as an indicator for how well the different interpolation methods can predict the actual shape of the curve. Linear interpolation gives the highest average difference, while spline interpolation gives a somewhat lower difference than polynomials. We also notice that a polynomial of degree 4 has a larger average error than a polynomial of degree 3. This may be partially dependent on the data set used here, but also related to Runge's phenomenon [6] of undesired effects when using higher-degree polynomials for curve fitting. In this case, spline interpolation is the best method to predict turbine efficiency between the measured points. Similar test procedures should be carried out for a larger number of real-life data sets to verify the general validity of these conclusions.

4.3. Optimal scheduling in the day-ahead market

We use a simple production system with one reservoir and one plant with a single generator to show how the different interpolation methods affect the results in STHS problem. The market price is taken from four days of historical prices on the Nord pool spot market in the beginning of 2018 [13]. The optimal discharge found by the optimization model is plotted in Figure 5, with linear interpolation and spline interpolation, respectively.

In this relatively small optimization problem, the calculation time is 1.21 seconds with linear interpolation and 1.38 seconds with spline interpolation. A consequence of the continuously varying gradient with spline interpolation is that the discharge changes more smoothly than with linear interpolation. The largest difference in discharge between the two interpolation methods for a single hour is $2.4 \text{ m}^3/\text{s}$, corresponding to nearly 10% of the optimal result. Note that the lowest discharge level is $25.42 \text{ m}^3/\text{s}$ with linear interpolation, whereas just above $26 \text{ m}^3/\text{s}$ with spline interpolation. The lowest discharge level is at the best efficiency point for the unit. In Table 1 we see that this point is at $25.42 \text{ m}^3/\text{s}$, and it remains at this level with linear interpolation. However, since we have not added a constraint saying $f'_n(q_n) = 0$ at the best point when finding the spline, the set of equations allows the best point to deviate from the one defined in Table 1. Adding such a constraint means that we have to remove one of the other constraints for the spline. If we do not remove another constraint, the system of equations will be overdetermined and a solution might not be found.

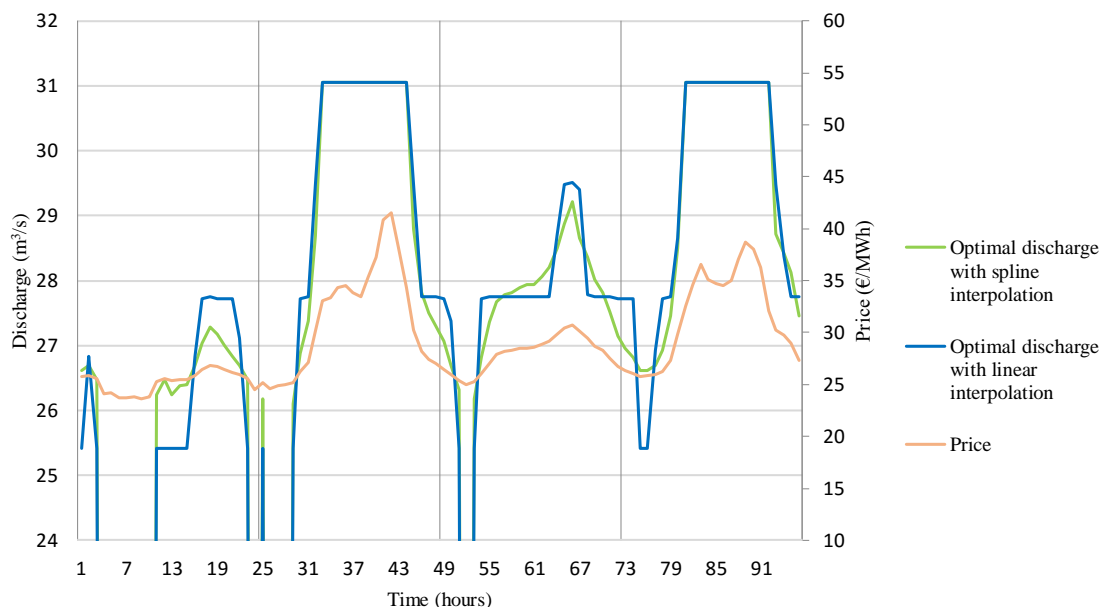


Figure 5. Discharge in the day-ahead market with linear and spline interpolation for turbine efficiency.

4.4. Calculation of marginal cost curves for intraday trading

In Figure 5 from the previous section there are four main discharge levels with linear interpolation, 25.42, 27.74, 29.5 and 31.06 m³/s. This set of discrete discharge levels is easy to explain if we look at the marginal cost curve in Figure 6. The marginal cost curve with linear interpolation has three vertical jumps at 27.74 m³/s, 29.5 m³/s and 31.06 m³/s, respectively. This means that all prices within each vertical jump will result in the same discharge. As concluded in Section 3.4, linear interpolation is not suitable to build bidding curves that should be continuous for both discharge and marginal cost. Nonlinear interpolation is necessary for creating continuous bidding curves for intraday markets. In this case, the largest difference in marginal cost for a given discharge is 1.7 €/MWh between the two interpolation methods. This highlights the importance of nonlinear spline interpolation for the calculation of the marginal cost curve.

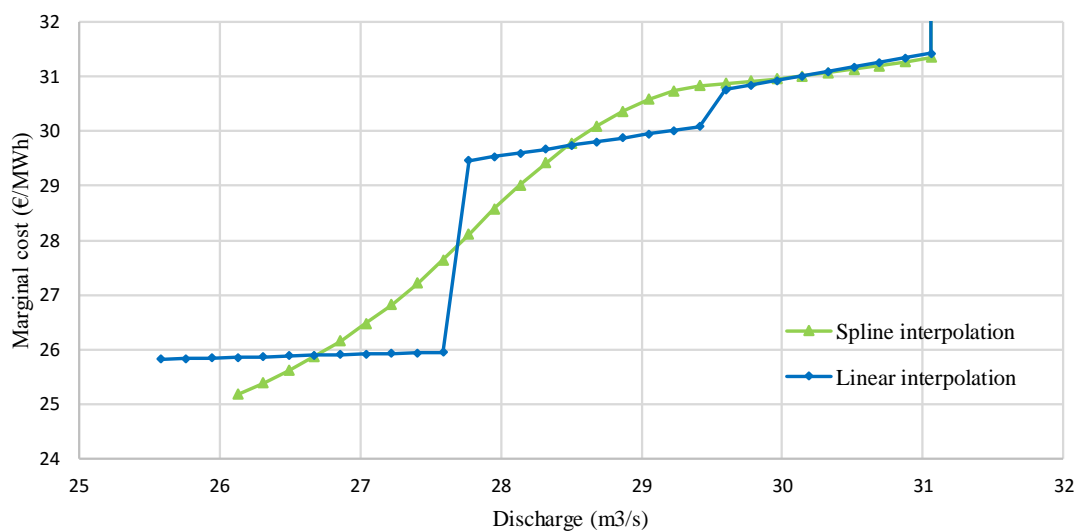


Figure 6. Resulting marginal cost curves with linear and spline interpolation.

5. Conclusion

In this paper we have identified both strengths and weaknesses of two interpolation methods for turbine efficiency curves. Linear interpolation makes it easier for the user to define the best operating point for a unit directly in the input data to the STHS problem, while spline interpolation gives the best estimate for efficiency for points that have not been measured. Spline interpolation also gives smoother discharge transitions in day-ahead scheduling and is crucial for obtaining continuous marginal cost curves in intraday trading. Since the tests are done only for one plant, further work should include a larger range of test systems to verify the validity of these conclusions. Based on the results in this paper, spline interpolation is a promising alternative to linear and polynomial interpolation for turbine efficiency curves in STHS.

References

- [1] Diniz A L, Esteves P P I and Sagastizabal C A 2007 A mathematical model for the efficiency curves of hydroelectric units *IEEE Power Engineering Society General Meeting, Vols 1-10* 200-6
- [2] Soares S and Salmazo C T 1997 Minimum loss predispach model for hydroelectric power systems *IEEE Transactions on Power Systems* **12** 1220-7

- [3] Sousa T, Jardini J A and de Lima R A 2010 In Site Hydroelectric Power Plant Unit Efficiency Measurement 2010 *IEEE PES Transmission and Distribution Conference and Exposition: Smart Solutions for a Changing World*, (New Orleans, LA)
- [4] Hidalgo I G, Fontane D G, Arabi M, Lopes J E G, Andrade J G P and Ribeiro L 2012 Evaluation of Optimization Algorithms to Adjust Efficiency Curves for Hydroelectric Generating Units *Journal of Energy Engineering-Asce* **138** 172-8
- [5] Hidalgo I G, Fontane D G, Lopes J E G, Andrade J G P and de Angelis A F 2014 Efficiency Curves for Hydroelectric Generating Units *Journal of Water Resources Planning and Management* **140** 86-91
- [6] Runge C 1901 Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten. *Zeitschrift für Mathematik und Physik* **246** 224-43
- [7] Aggidis G A and Zidonis A 2014 Hydro turbine prototype testing and generation of performance curves: Fully automated approach *Renewable Energy* **71** 433-41
- [8] Aradag S, Akin H and Celebioglu K 2017 CFD based design of a 4.3MW Francis turbine for improved performance at design and off-design conditions *Journal of Mechanical Science and Technology* **31** 5041-9
- [9] International Electrotechnical Commission 1999 Standard 60193 Hydraulic turbines, storage pumps and pump-turbines - Model acceptance tests
- [10] Skjelbred H I, Kong J, Belsnes M M and Fosso O B 2018 Comprehensive nonlinear modelling of hydropower plants in short-term hydro scheduling. SINTEF Energy Research
- [11] Skjelbred H I, Kong J, Larsen T J and Kristiansen F 2017 Operational use of marginal cost curves for hydropower plants as decision support in real-time balancing markets. In: *14th International Conference on the European Energy Market - EEM 2017*, (Dresden, GERMANY)
- [12] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 1992 *Numerical Recipes in C*. Cambridge University Press
- [13] <https://www.nordpoolgroup.com/> Nord Pool