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On the Globally Exponentially Convergent Immersion and Invariance Speed Observer for Mechanical Systems

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***Abstract*—In this article we present a reformulation of the invariance and immersion speed observer of Astolfi et al. as applied to mechanical systems with bounded inertia matrices. This is done to explore the possibility of its practical implementation e.g. for 6 degrees-of-freedom industrial robots. The reformulation allows us find an explicit expression for one of the bounds used in the observer, and a constructive method for the second. We show that the observer requires either analytically or numerically solving at most $2n^2$ integrals, where n is the number of generalized coordinates in the mechanical system.**

I. Introduction

Industrial robots are typically equipped with encoders to measure the angles of the joints. These give accurate and steady measurements of the angles, but the speeds of these angles are not always available. Some systems have a tachometer, inertial sensors, or perform numerical differentiation of the joint position. These methods can introduce high-frequency noise or phase lag. An alternative method is to construct an observer for the nonlinear system.

Speed observers have been an important topic in robotics. One of the first references to a speed observer for mechanical systems is [1], where an asymptotic observer was used in a feedback situation with a trajectory tracking controller. In [2] a semi-globally exponentially stable observer based on passivity was presented. In [3] a globally exponentially convergent observer was established for general Euler-Lagrange systems. In [4] the observer was applied to systems with non-holonomic constraints and the system written in port-Hamiltonian form. In [5], a novel observer was presented with globally exponentially stable properties given that the inertia matrix has an upper bound. Speed observers are most notably featured in the topic of output-feedback control, where there are two main approaches: model-based approaches which utilize speed observers [5], [6], and filter-based approaches which use filters to replace speed observers [7]. In this article we are interested in the speed observer of [3]. The usage of the invariance and immersion observer for the output-feedback tracking scenario is outside the scope of this article, however we refer the reader to [8] where a globally exponentially stable trajectory controller that uses the immersion and invariance observer for a mechanical system in port-Hamiltonian form is presented.

In [9], Karagiannis et al. showed that a globally asymptotically convergent speed observer can be constructed for 2 degrees-of-freedom mechanical system if a certain partial differential equation admits analytical solutions. In [10] Karagiannis et al. presents a method of approximating such a partial differential equation using output filters and a dynamic scaling parameter. The deviation between the partial derivative of the approximated solution and the ideal solution is compensated for by the dynamic scaling parameter. In [3], Astolfi et al. applied this approximation method to general Euler-Lagrange systems to create a globally exponentially convergent speed observer. Astolfi et al. showed that there exist some bounds on the disturbances introduced by the deviation from the ideal partial differential equation. The speed observer was to be considered a proof of existence rather than a directly implementable method. The reason for this is that it requires the solution of a set of integrals that may not have closed-form solutions. To that end, they must be approximated numerically. Future developments in computational power may allow us to perform the necessary numerical integrations on-line. To that effect this article establishes how many integrals are needed at most. Given the Euler-Lagrange equations, the remaining difficulty is then how to define the necessary bounds on our deviation from the ideal partial differential equation.

In both [9] and [3], the speeds of the mechanical system were transformed by the Cholesky factorization of the inertia matrix, see the preliminary lemma of [3]. This transformation gives rise to a skew-symmetric property that simplifies the Lyapunov analysis. The basis of which stems from the skew-symmetry of the derivative of the inertia matrix. A similar transformation is performed on the port-Hamiltonian system in [4]. In this paper we show that, given the property that the inertia matrix has an upper bound, this transformation is not necessary. This excludes us from applying the observer to systems with an infinitely extending prismatic joint, but allows us to make an explicit bound on one of the disturbances. We give a constructive method of finding the other bound, a method that can also be applied to the observer of Astolfi et al., and we show how a naively implemented observer requires the evaluation of at most $2n^2$ integrals. This article is an attempt to explore the possibility of using the invariance and immersion globally exponentially convergent observer for general mechanical robots with bounded inertia matrices.

The paper is organized as follows: Section II contains two subsections, system and observer. The system subsection describes the system discussed in this paper and the properties assumed for it, and the observer subsection follows the proof of stability of our observer, a parallel to the proof of [3], and presents two lemmas for finding the bounds. Section III contains two sections, system description and results. Finally, Section IV contains the discussion and conclusion.

II. Theory

A. System

In this paper we consider an n degrees-of-freedom robot described by the differential equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \tag{1}$$

where $q \in \mathbb{R}^n$ are the generalized coordinates, $M(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the Coriolis matrix, $G(q) \in \mathbb{R}^n$ is a vector of potential forces, and $u \in \mathbb{R}^n$ is the control input. For any $q, x, y \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ we assume the following properties hold

P1: $k_M I \geq M(q) \geq k_m I$
P2: $C(q,x)y = C(q,y)x$
P3: $C(q,\lambda_1 x_1 + \lambda_2 x_2)y = \lambda_1 C(q,x_1)y + \lambda_2 C(q,x_2)y$
P4: $\|C(q,x)\| \leq k_c \|x\|$
P5: $\dot{M}(q) = C(q,\dot{q}) + C(q,\dot{q})^T$.

with some known positive k_M, k_m, and k_c. The property P4 is not required for stability, but simplifies one of the bounds as will be shown in Lemma 1. P5 is to cancel the Coriolis matrices, for more information on applicability of P1 and P5 we refer the reader to [11]. The system admits the state-space representation:

$$\dot{y} = x \tag{2a}$$
$$\dot{x} = M(y)^{-1}(u - C(y,x)x - G(y)) \tag{2b}$$

with y the measured coordinates, and x being the unmeasured speeds of the generalized coordinates.

B. Observer

For the system (2), we define the observer

$$\dot{\xi} = \alpha_1(\xi, s_y, s_x, y, u, r) \tag{3a}$$
$$\hat{x} = \xi + \beta(y, s_y, s_x) \tag{3b}$$
$$\dot{s}_y = \alpha_2(\hat{x}, s_y, s_x, y, r) \tag{3c}$$
$$\dot{s}_x = \alpha_3(\hat{x}, s_y, s_x, y, u, r) \tag{3d}$$
$$\dot{r} = \alpha_4(\hat{x}, s_y, s_x, y, r) \tag{3e}$$

where ξ, $\hat{x}$, s_y, s_x and r are the internal states of the observer. The output of the observer will be s_x, a filtered version of $\hat{x}$. We will define the dynamics α_i and mapping β will so that the error

$$z = \hat{x} - x \tag{4}$$

has a globally exponentially stable origin. The proof of this follows the proof in [3], except that we use P5 instead of the preliminary lemma of [3], where x in (2b) is transformed by the Cholesky factorization of the inertia matrix. For brevity we will omit the arguments of α_i when referred to in the following proof.

The dynamics of the error is found from (4) and taking the derivative of (3b) as

$$\begin{aligned}\dot{z} =& \alpha_1 + \frac{\partial\beta}{\partial y}x + \frac{\partial\beta}{\partial s_y}\alpha_2 + \frac{\partial\beta}{\partial s_x}\alpha_3 \\ &- M(y)^{-1}\left(u - C(y,x)x - G(y)\right)\end{aligned} \tag{5}$$

choosing α_1 as

$$\begin{aligned}\alpha_1 =& M(y)^{-1}\left(u - C(y,\hat{x})\hat{x} - G(y)\right) \\ &- \frac{\partial\beta}{\partial y}\hat{x} - \frac{\partial\beta}{\partial s_y}\alpha_2 - \frac{\partial\beta}{\partial s_x}\alpha_3\end{aligned} \tag{6}$$

the estimation error dynamics becomes

$$\dot{z} = M(y)^{-1}\left(C(y,x)x - C(y,\hat{x})\hat{x}\right) - \frac{\partial\beta}{\partial y}z. \tag{7}$$

From P2 and adding and subtracting $C(y,x)\hat{x}$, we have

$$C(y,x)x - C(y,\hat{x})\hat{x} = -C(y,x)z - C(y,\hat{x})z \tag{8}$$

giving the dynamics

$$\dot{z} = -M(y)^{-1}\left(C(y,x) + C(y,\hat{x})\right)z - \frac{\partial\beta}{\partial y}z. \tag{9}$$

With a Lyapunov function $V(t,z) = \frac{1}{2}z^T M(y)z$, and using P5, we can see that solving $\frac{\partial\beta}{\partial y}$ such that

$$\frac{\partial\beta}{\partial y} = M(y)^{-1}\left(k_1 I - C(y,\hat{x})\right) \tag{10}$$

yields a globally exponentially convergent observer for $k_1 > 0$. But the partial differential equation is not always solvable, we follow the proof of [3] and approximate (10).

We define

$$H(y,\hat{x}) \coloneqq M(y)^{-1}\left(k_1 I - C(y,\hat{x})\right) \tag{11}$$

as the ideal we want. Then following step 2 in [3], which uses a method from Karagiannis et al. in [10], we choose to model β as the sum of n integrals

$$\begin{aligned}\beta(y,s_y,s_x) \coloneqq& \int_0^{y_1(t)} H_1([y_1, s_{y2}, \ldots, s_{yn}], s_x)\mathrm{d}y_1 + \ldots \\ &+ \int_0^{y_n(t)} H_n([s_{y1}, \ldots, s_{yn-1}, y_n], s_x)\mathrm{d}y_n\end{aligned} \tag{12}$$

where the subscript i of H_i is the ith column of the matrix H, and s_{yi}, y_i refers to the ith element of the vectors. Note that we are substituting $\hat{x}$ for the filtered state s_x, and y for the filtered state s_y for all but the integrated elements y_i. We will denote this as the column H_i being a function of y_i, $s_{y1:n\setminus i}$, and s_x. This gives us the partial differential equation

$$\frac{\partial \beta}{\partial y} = \left[H_1([y_1, \dots, s_{yn}], s_x) \dots H_n([s_{y1}, \dots, y_n], s_x)\right]. \tag{13}$$

We can now also find the partial derivatives needed for α_1, starting with

$$\frac{\partial \beta}{\partial s_{yi}} = \sum_{j=0, j\neq i}^{n} \int_0^{y_j(t)} \frac{\partial H_j}{\partial s_{yi}}(y_j, s_{y1:n\setminus j}, s_x) \mathrm{d}y_j \tag{14}$$

which requires at most we $n(n-1)$ integrals. Secondly we have

$$\frac{\partial \beta}{\partial s_{xi}} = \sum_{j=0}^{n} \int_0^{y_j(t)} \frac{\partial H_j}{\partial s_{xi}}(y_j, s_{y1:n\setminus j}, s_x) \mathrm{d}y_j \tag{15}$$

similarly to (14), it requires evaluating n^2 integrals.

We define the deviation of our modeled β from the ideal as

$$\begin{aligned} &H_i(y, \hat{x}) - H_i(y_i, s_{y1:n\setminus i}, s_x) = \\ &\quad H_i(y, \hat{x}) - H_i(y, s_x) + H_i(y, s_x) - H_i(y_i, s_{y1:n\setminus i}, s_x) \\ &\quad = \Delta_{x,i}(y, s_x, e_x) + \Delta_{y,i}(y, s_x, e_y) \end{aligned} \tag{16}$$

where i refers to the column vector of the matrices Δ_x and Δ_y, and we have defined the errors

$$e_y := s_y - y \tag{17}$$

$$e_x := s_x - \hat{x} \tag{18}$$

which, from (13) and (16), gives us

$$\frac{\partial \beta}{\partial y} = H(y, \hat{x}) - \Delta_x(y, s_x, e_x) - \Delta_y(y, s_y, e_y). \tag{19}$$

As (13) is the same as (11) when $y = s_y$ and $\hat{x} = s_x$, we see that when $e_y = 0$ and $e_x = 0$ then $\Delta_x = 0$ and $\Delta_y = 0$. In combination with the fact that the mappings are smooth this ensures that there exists some mappings $\bar{\Delta}_x$, $\bar{\Delta}_y$ such that

$$\Delta_x(y, s_x, e_x) = \bar{\Delta}_x(y, s_x, e_x)e_x \tag{20a}$$

$$\Delta_y(y, s_y, e_y) = \bar{\Delta}_y(y, s_y, e_y)e_y. \tag{20b}$$

Theoretically we can see that $\bar{\Delta}_x$ and $\bar{\Delta}_y$ can be found by taking the Taylor series around $e_x = 0$ and $e_y = 0$ and factoring out e_x or e_y. But as many systems contain trigonometric expressions, this is potentially an infinite series that is not easy to implement. The following lemma gives a constructive method of finding $\|\Delta_x\|$ related to $\|e_x\|$.

Lemma 1. *Given a matrix $\Delta_x(y, s_x, e_x)$ as defined in (16), and the properties* P1, P3, *and* P4, *then*

$$\|\Delta_x(y, s_x, e_x)\| \leq \frac{k_c}{k_m} \|e_x\| \tag{21}$$

Proof. From (16) we have

$$\begin{aligned} \Delta_x(y, s_x, e_x) &= H(y, s_x - e_x) - H(y, s_x) \\ &= M(y)^{-1}\left(-C(y, s_x - e_x) + C(y, s_x)\right) \\ &= M(y)^{-1}\left(C(y, e_x)\right) \end{aligned} \tag{22}$$

where we have used P3 to arrive at the last line. And thus

$$\|\Delta_x(y, s_x, e_x)\| \leq \left\|M(y)^{-1}\right\| \|C(y, e_x)\| \tag{23}$$

which, given P1 and P4 becomes (21). □

When P4 is available, Lemma 1 gives an explicit bound on Δ_x. For Δ_y the partial substitution of s_y for y for all but element y_i complicates matters. The following lemma is more general and is used to find bounds that relate $\|\Delta_x\|$ or $\|\Delta_y\|$ to $\|e_x\|$ or $\|e_y\|$.

Lemma 2. *Given a function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ that is continuously differentiable, where*

$$\Delta(x, y, e) = 0 \tag{24}$$

if

$$e = 0 \tag{25}$$

and

$$\left\| \sup_{e\in\mathbb{R}^n} \frac{\partial \Delta_{ij}(x, y, e)}{\partial e} \right\|_2 = \bar{\Delta}_{ij}(x, y, e) \tag{26}$$

where the subscript ij refers to an element in the matrix. Then

$$\|\Delta(x, y, e)\|_2 \leq \left\|\bar{\Delta}(x, y, e)\right\|_F \|e\|_2 \tag{27}$$

Proof. From the definition of the matrix 2 norm and Frobenius norm we have

$$\|\Delta(x, y, e)\|_2 \leq \|\Delta(x, y, e)\|_F := \tag{28}$$

$$sqrt \sum_{i=1}^{m} \sum_{j=1}^{m} \Delta_{ij}(x, y, e)^2 \tag{29}$$

from (26), there exists a supremum such that

$$\begin{aligned} |\Delta_{ij}(x, y, e)| &\leq \left| \left(\sup_{e\in\mathbb{R}^n} \frac{\partial \Delta_{ij}(x, y, e)}{\partial e} \right) e \right| \\ &\leq \bar{\Delta}_{ij}(x, y, e) \|e\|_2 \end{aligned} \tag{30}$$

and putting this into the (29) gives

$$\|\Delta(x, y, e)\|_2 \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} \bar{\Delta}_{ij}(x, y, e)^2} \|e\|_2 \tag{31}$$

which is equal to (27). □

Continuing on the stability proof, we substitute (19) in (9)

$$\begin{aligned} \dot{z} = &- M(y)^{-1}\left(C(y, x) + C(y, \hat{x})\right) z \\ &- H(y, \hat{x})z + (\Delta_y + \Delta_x)z \end{aligned} \tag{32}$$

and using (11) we get

$$\dot{z} = -M(y)^{-1}\left(k_1 I + C(y, x)\right) z + (\Delta_y + \Delta_y)z \tag{33}$$

The matrices Δ_x and Δ_y act as disturbances on z that we will dominate with a dynamic scaling r. We define a new scaled variable as

$$\eta = \frac{z}{r} \tag{34}$$

with the derivative

$$\dot{\eta} = \frac{\dot{z}}{r} - \frac{\dot{r}}{r}\eta. \tag{35}$$

We define the Lyapunov function $V_1(t,\eta) = \frac{1}{2}\eta^T M(y)\eta$ so as to cancel the Coriolis matrices. Using (33), (35), and P5, we get

$$\dot{V}_1 = -k_1\|\eta\|^2 + \eta^T M(y)(\Delta_y + \Delta_x)\eta - \frac{\dot{r}}{r}\eta^T M(y)\eta \quad (36)$$

using P1

$$\begin{aligned}\dot{V}_1 &\leq -k_1\|\eta\|^2 + \|M(y)(\Delta_y+\Delta_x)\|\,\|\eta\|^2 - \frac{\dot{r}}{r}k_m\|\eta\|^2 \\ &\leq -\left(\frac{k_1}{2} + k_m\frac{\dot{r}}{r} - \frac{1}{2k_1}\|M(y)(\Delta_y+\Delta_x)\|^2\right)\|\eta\|^2 \\ &\leq -\left(\frac{k_1}{2} + k_m\frac{\dot{r}}{r} - \frac{1}{k_1}(\|M(y)\Delta_y\|^2 + \|M(y)\Delta_x\|^2\right)\|\eta\|^2\end{aligned} \quad (37)$$

where the second inequality is found using Young's inequality with factor k_1. Choosing the dynamics of r as

$$\dot{r} = -\frac{k_1}{4k_m}(r-c_r) + \frac{r}{k_1k_m}\left(\|M(y)\Delta_y\|^2 + \|M(y)\Delta_x\|^2\right) \quad (38)$$

with $r(t_0) > c_r > 0$ and $r(t) > c_r > 0$ gives

$$\dot{V}_1 \leq -\left(\frac{k_1}{2} - \frac{k_1}{4}\frac{r-c_r}{r}\right)\|\eta\|^2 \leq -\frac{k_1}{4}\|\eta\|^2. \quad (39)$$

This gives global exponential stability of $\eta(t)$, and in turn we need boundedness of $r(t)$ to ensure global exponential stability of $z(t)$. The choice of placing a parameter c_r as a lower bound on r is inspired by [5].

We are going to create the Lyapunov functions

$$V_2(t,\eta,e_y,e_x) = V_1(t,\eta) + \frac{1}{2}(e_y^T e_y + e_x^T e_x) \quad (40)$$

$$V_3(t,\eta,e_y,e_x,r) = V_2(t,\eta,e_y,e_x) + \frac{1}{2}(r-c_r)^2. \quad (41)$$

From the definition of the errors (17) and (18), and taking the derivative of (3b), we have

$$\dot{e}_y = \alpha_2 - x \quad (42)$$

$$\dot{e}_x = \alpha_3 - \alpha_1 + \frac{\partial\beta}{\partial y}x + \frac{\partial\beta}{\partial s_y}\alpha_2 + \frac{\partial\beta}{\partial s_x}\alpha_3 \quad (43)$$

choosing

$$\alpha_2 = \hat{x} - \psi_1 e_y \quad (44)$$

$$\alpha_3 = M(y)^{-1}(u - C(y,\hat{x})\hat{x} - G(y)) - \psi_2 e_x \quad (45)$$

where ψ_1 and ψ_2 are scalar gain functions, with our chosen α_1, (6), we have

$$\dot{e}_y = z - \psi_1 e_y \quad (46)$$

$$\dot{e}_x = \frac{\partial\beta}{\partial y}z - \psi_2 e_x. \quad (47)$$

The time derivative of V_2 is therefore

$$\dot{V}_2 = \dot{V}_1 + re_y^T\eta + re_x^T\frac{\partial\beta}{\partial y}\eta - \psi_1 e_y^T e_y - \psi_2 e_x^T e_x \quad (48)$$

$$\begin{aligned}&\leq -\left(\frac{k_1}{4} - 1\right)\|\eta\|^2 - \left(\psi_1 - \frac{r^2}{2}\right)\|e_y\|^2 \\ &\quad - \left(\psi_2 - \frac{r^2}{2}\left\|\frac{\partial\beta}{\partial y}\right\|^2\right)\|e_x\|^2\end{aligned} \quad (49)$$

where we have used Young's inequality. From (38), we get the time derivative of V_3 as

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 - \frac{k_1}{4k_m}(r-c_r)^2 \\ &\quad + \frac{r(r-c_r)}{k_1k_m}\left(\|M(y)\Delta_y\|^2 + \|M(y)\Delta_x\|^2\right).\end{aligned} \quad (50)$$

As we have established (20a) and (20b), or similarly appropriate bounds through Lemma 1 and Lemma 2, we arrive at the inequality

$$\begin{aligned}\dot{V}_3 \leq &\dot{V}_2 - \frac{k_1}{4k_m}(r-c_r)^2 \\ &+ \frac{r^2k_M^2}{k_1k_m}\left(\|\bar{\Delta}_y\|^2\|e_y\|^2 + \|\bar{\Delta}_x\|^2\|e_x\|^2\right)\end{aligned} \quad (51)$$

where we have used the fact that $r^2 > r(r-c_r)$ for $r > c_r > 0$. Collecting the terms from (49) we get

$$\begin{aligned}\dot{V}_3 \leq &-\left(\frac{k_1}{4} - 1\right)\|\eta\|^2 - \left(\psi_1 - \frac{r^2}{2} - \frac{r^2k_M^2}{k_1k_m}\|\bar{\Delta}_y\|^2\right)\|e_y\|^2 \\ &- \left(\psi_2 - \frac{r^2}{2}\left\|\frac{\partial\beta}{\partial y}\right\|^2 - \frac{r^2k_M^2}{k_1k_m}\|\bar{\Delta}_x\|^2\right)\|e_x\|^2 \\ &- \frac{k_1}{4k_m}(r-c_r)^2\end{aligned} \quad (52)$$

we can see that if we choose

$$\psi_1 = k_2 + \frac{r^2}{2} + \frac{r^2k_M^2}{k_1k_m}\|\bar{\Delta}_y\|^2 \quad (53)$$

$$\psi_2 = k_3 + \frac{r^2}{2}\left\|\frac{\partial\beta}{\partial y}\right\|^2 + \frac{r^2k_M^2}{k_1k_m}\|\bar{\Delta}_x\|^2 \quad (54)$$

we get

$$\begin{aligned}\dot{V}_3 \leq &-\left(\frac{k_1}{4} - 1\right)\|\eta\|^2 - k_2\|e_y\|^2 - k_3\|e_x\|^2 \\ &- \frac{k_1}{4k_m}(r-c_r)^2\end{aligned} \quad (55)$$

where we choose $k_1 > 4$. This gives $\dot{V}_3 \leq 0$, which ensures that $r(t)$ is bounded. This in turn gives us global exponential convergence of $z(t)$.

As shown in [9], the difference between the solution to the ideal partial differential equation (10) and our approximation (12) is such that the states from the output filters, s_y and s_x give the best estimates of the system states from the "perspective" of our observer. One way to think of this is to see that ξ would have been the states of our ideal observer, with output defined by $\hat{x}$. We cannot solve the differential equation required to render β of the output function as we want it. So we give it dynamics s_x and s_y that filter the states to compensate for the disturbances introduced by the approximation. This means that s_y, and s_x are the output of our observer.

III. SIMULATION

A. System description

We consider the 2 degrees-of-freedom system used in [3], with y_1 and y_2 defined as illustrated in Fig.1. We define

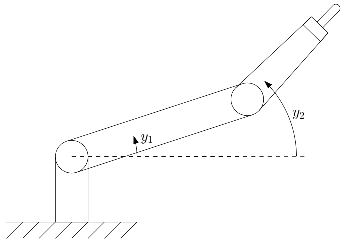

Fig. 1. Two-link manipulator with revolute joints.

$c(x,y) = \cos(x-y)$, $s(x,y) = \sin(x-y)$, $D(x,y) = c_1c_2 - c_3^2 c(x,y)^2$, and the system is described by

$$M(y) = \begin{bmatrix} c_1 & c_3 c(y_1,y_2) \\ c_3 c(y_1,y_2) & c_2 \end{bmatrix} \tag{56a}$$

$$C(y,x) = \begin{bmatrix} 0 & -c_3 s(y_1,y_2)x_1 \\ c_3 s(y_1,y_2)x_2 & 0 \end{bmatrix} \tag{56b}$$

$$G(y) = g \begin{bmatrix} c_4 \cos(y_1) \\ c_5 \cos(y_2) \end{bmatrix} \tag{56c}$$

with

$$c_1 = I_1 + I_2 + m_1 L_{c_1}^2 + m_2(L_1^2 + L_{c_2}^2) \tag{57a}$$

$$c_2 = 2m_2 L_{c_2} L_1,\ c_3 = m_2 L_{c_2}^2 + I_2 \tag{57b}$$

$$c_4 = g(L_{c_1}(m_1 + M_1) + L_1(m_2 + M_2))g \tag{57c}$$

$$c_5 = g L_{c_2}(m_2 + M_2) \tag{57d}$$

where the links are of length L_i, with link masses M_i at L_{c_i}, and masses m_i at the joints. We have

$$k_m = \lambda_{\text{min}}(M_0)), \quad k_M = \lambda_{\text{max}}(M_0), \quad k_c = c_3 \tag{58}$$

where k_m and k_M stem from the eigenvalues λ of M_0, the inertia matrix M at the origin $[y_1, y_2] = [0,0]$, and using the method of [12] we get k_C.

Using (11) we have:

$$H(y,x) = \begin{bmatrix} \frac{c_2k_1 + \frac{c_3^2 x_2}{2} s(2y_1,2y_2)}{D(y_1,y_2)} & \frac{c_3(c_2x_1 s(y_1,y_2) - k_1 c(y_1,y_2))}{D(y_1,y_2)} \\ -\frac{c_3(c_1x_2 s(y_1,y_2) + k_1 c(y_1,y_2))}{D(y_1,y_2)} & \frac{c_1k_1 - \frac{c_3^2 x_1}{2} s(2y_1,2y_2)}{D(y_1,y_2)} \end{bmatrix} \tag{59}$$

from which Δ_x, and Δ_y can be constructed using (16). A simple albeit overestimating method of finding $\bar{\Delta}_{ij}$ from a symbolically calculated $\frac{\partial \Delta_{ij}}{\partial e}$ is the sum of the supremum of the terms of the expression. E.g. if $\frac{\partial \Delta_{ij}}{\partial e} = f_1 - f_2 + f_3$ then we choose $\bar{\Delta}_{ij} = \|f_1\|_\infty + \|f_2\|_\infty + \|f_3\|_\infty$. The initial states of the robot and the observer are given in Table I, and the robot parameters are given in Table II. The initial states of the robot and the initial estimate s_x were chosen so as to coincide with the initial states in [3].

B. Results

The graphs show the observer of this paper with parameters specified in Table II as well as that of [3] with $k_1 = 10$. In Fig.2 we see that the filtered angles s_y converge to the actual trajectories y over time. In Fig.3 we can see that the filtered speeds s_x converge to the actual speeds x over time. Note that with the same gains as in [3] our observer takes a longer time to converge. It appears that we require larger gains to achieve the same effect as in [3]. In Fig.5 we attempt to show our reasoning for choosing a small initial $r(t_0)$, and using c_r to allow an r lower than 1. The graph shows the norm of $[\dot{\xi}, \dot{s}_y, \dot{s}_x, \dot{r}]$ evaluated for varying r at the initial state. With larger r the observer differential equation is stiffer, requiring a small timestep to remain accurate. This trend was observed for many states other than the initial state as well. To simplify simulation, we chose a sufficiently small $r(t_0)$ so as to remain in the region where $r < 10^{-1}$.

TABLE I
INITIAL STATES OF OBSERVER AND ROBOT

$y(0)$	$x(0)$	$\xi(0)$	$s_y(0)$	$s_x(0)$	$r(0)$
$[0,0]^T$	$[-0.29, 6.66]^T$	$[-8.45, 68.33]^T$	$[1,2]^T$	$[10,20]^T$	0.1

TABLE II
ROBOT AND OBSERVER PARAMETERS

c_1	c_2	c_3	c_4	c_5	k_1	k_2	k_3	c_r
0.9698	0.1575	0.264	1.74	0.44	40	8	2.5	0.001

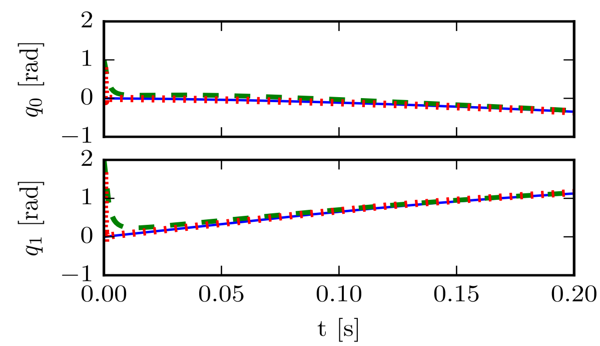

Fig. 2. Joint angles over time. The blue line is the actual angles, the dashed green line is s_y, and the dotted red line is the filtered angles from the observer in [3]. Note that this is shown for the timeframe 0 to 0.2 s.

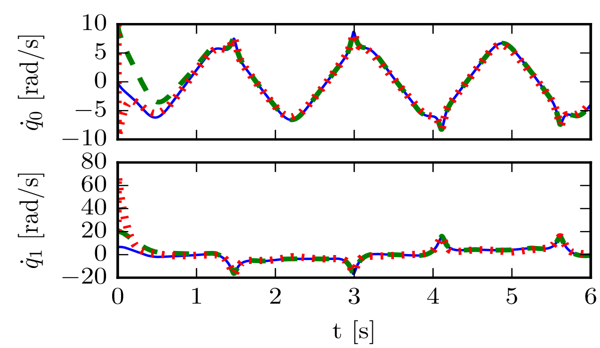

Fig. 3. Joint speeds over time. The blue line is the actual speeds, the dashed green line is s_x, and the dotted red line is the estimated speeds of the observer in [3].

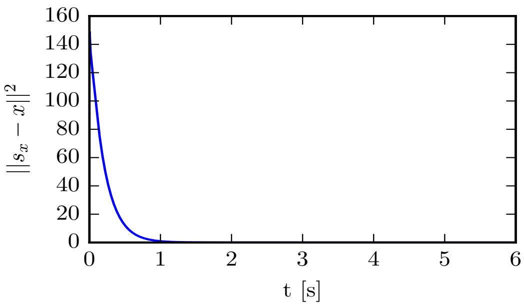

Fig. 4. $\|s_x - x\|^2$ over time.

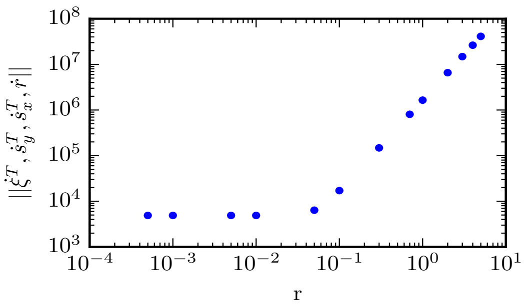

Fig. 5. The norm of $[\dot{\xi}^T, \dot{s}_y^T, \dot{s}_x^T, \dot{r}]^T$ with respect to r. Evaluated at the initial state of the observer described in Table I.

IV. Discussion and Conclusion

A. Discussion

Our alternative formulation of the observer does not come for free, we use a Lyapunov function V_1 that depends on the inertia matrix. This means that unlike [3] we require an upper bound on the inertia matrix, a property that does not hold for mechanical systems with inifinitely extending prismatic joints. Further work into relating the observer presented in this article to that of [3] may give explicit bounds for the disturbances. Lemma 2 can also be used for the disturbance bounds in [3].

As the second Lemma relies on finding the supremums of nonlinear equations, it is not as practical as Lemma 1. Investigating the general structure of Δ_y with respect to some classes of robotic systems, e.g. consisting of revolute joints defined by the Denavit-Hartenberg convention, might give rise to explicit bounds.

We can see that if all the integrals of (12) are performed numerically, which would allow us to generate an observer from only the symbolic system equations, naively implemented, we will end up with $2n^2$ numerical integrals from our approximation of β, $\frac{\partial \beta}{\partial s_y}$, and $\frac{\partial \beta}{\partial s_x}$ evaluated at each timestep. This might be cumbersome even with optimized methods of performing the integrals and parallelising the effort. Further work evaluating what to do when parts of the differential equation is solvable may reduce the number of numeric integrals needed. The exact number of integrals required depends on how one defines the generalized coordinates of the robot, and the mechanical structure of the system. For example, if the second joint in our example was defined relative to the first joint angle instead of the horizontal line, $H(y, \hat{x})$ would not depend on y_1 and the corresponding integrals would be zero.

B. Conclusion

In this article we presented the observer of Astolfi et al. from [3] reformulated so as to give more intuitive internal states. These make it easier to provide an explicit method in Lemma 1 and constructive method in Lemma 2 for defining the necessary bounds. The method of Lemma 2 is sufficiently general to be applied to the observer of [3]. A result of our reformulation of the observer is that it requires an upper bound on the inertia matrix, a property that Astolfi et al. did not require. Through our approximation of a partial differential equation, the observer requires evaluation of $2n^2$ integrals.

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