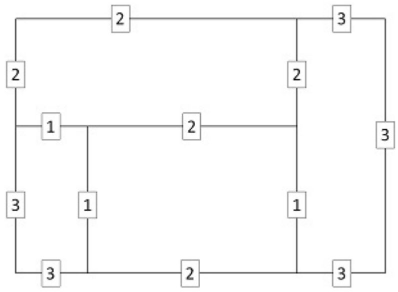
SINTEF A23875- Unrestricted

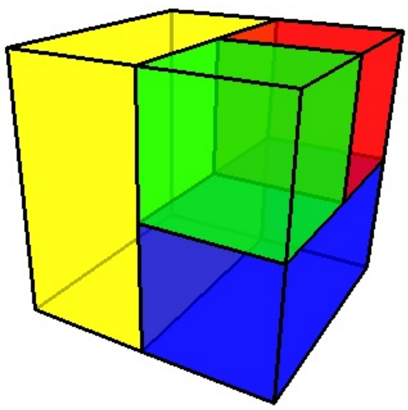
Report

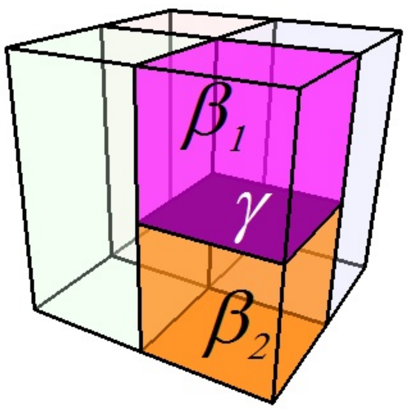
On the dimension of multivariate spline spaces

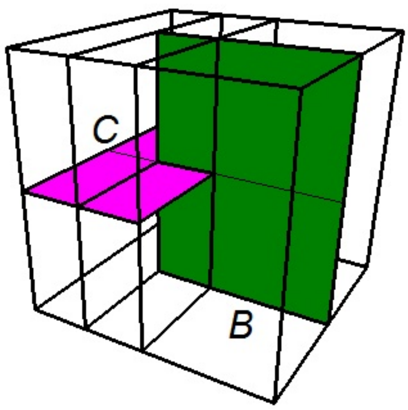
Preprint

Author(s)
Kjell Fredrik Pettersen









SINTEF ICT
Applied Mathematics
2013-01-09

SINTEF IKT
SINTEF ICT

Address:
Postboks 124 Blindern
NO-0314 Oslo
NORWAY

Telephone:+47 73593000
Telefax:+47 22067350

postmottak.IKT@sintef.no
www.sintef.no
Enterprise /VAT No:
NO 948 007 029 MVA

Report

On the dimension of multivariate spline spaces

Preprint

KEYWORDS:
Box-partitions
Locally refined B-splines
Spline spaces
Homology
Spline dimension
Multivariate splines

VERSION
1.0

DATE
2013-01-09

AUTHOR(S)
Kjell Fredrik Pettersen

CLIENT(S)
SINTEF

CLIENT'S REF.

PROJECT NO.
90511201

NUMBER OF PAGES/APPENDICES:
57

ABSTRACT

Abstract heading

In this paper, we define the topological structures for an arbitrary axis-aligned box partition of a parametric d-dimensional box-shaped limited domain in $\mathbf{R}^d$. Then we define the d-variate spline space over this partition with given polynomial degrees and arbitrary continuity constraints. We then use homological techniques to show that the dimension of this spline space can be split up as $\dim \mathbf{S}(N) = C + H$, where the first term is a combinatorial easily calculated term that only depends on the topological structure, polynomial degrees and continuity constraints, while the second term is an alternating sum of dimensions of homological terms. They are often zero, but not always, and might even in some special situations depend on the parameterization.

We give explicit expressions for the terms in tensor product spaces, before we look at how the homology modules are tied together during a refinement process. Eventually we discuss the cases d=2 and d=3.

PREPARED BY
Kjell Fredrik Pettersen

SIGNATURE

CHECKED BY
Tor Dokken

SIGNATURE

APPROVED BY
Trond Runar Hagen

SIGNATURE

REPORT NO.	ISBN	CLASSIFICATION	CLASSIFICATION THIS PAGE
SINTEF A23875	978-82-14-05313-5	Unrestricted	Unrestricted

SINTEF

Document history

VERSION	DATE	VERSION DESCRIPTION
1.	2013-01-09	Preprint version for web page publication

SINTEF

Table of contents

On the dimension of spline spaces in $\mathbb{R}^d$

Kjell Fredrik Pettersen*

January 10, 2013

Abstract

In this paper, we define the topological structures for an arbitrary axis-aligned box partition of a parametric d-dimensional box-shaped limited domain in $\mathbb{R}^d$. Then we define the d-variate spline space over this partition with given polynomial degrees and arbitrary continuity constraints. We then use homological techniques to show that the dimension of this spline space can be split up as $\dim \mathbb{S}(\mathcal{N}) = C + H$, where the first term is a combinatorial easily calculated term that only depends on the topological structure, polynomial degrees and continuity constraints, while the second term is an alternating sum of dimensions of homological terms. They are often zero, but not always, and might even in some special situations depend on the parameterization.

We give explicit expressions for the terms in tensor product spaces, before we look at how the homology modules are tied together during a refinement process. Eventually we discuss the cases $d = 2$ and $d = 3$.

1 Introduction

The purpose og this document is to give an overview of the results regarding the dimension formulas for spline spaces over non-regular axis-aligned partitions of rectangular domains in $\mathbb{R}^d$ for arbitrary d. The document is still to be considered as an unfinished draft, with no proof reading or quality checks.

*Kjell Fredrik Pettersen, SINTEF, PO Box 124 Blindern, 0314 Oslo, Norway, *email: KjellFredrik.Pettersen@sintef.no*

The work is a generalization of the dimension formula in [3], and is part of a joint work with Bernard Mourrin at INRIA, Nice, France. The dimension formula in [3] gives the dimension of the spline space of bivariate functions over a non-regular axis-aligned domain partition in $\mathbb{R}^2$. In this document, the formula is generalized to higher dimensions, to more general and varying continuity constraints between adjacent elements, and to non-open meshes.

The dimension formulas in this paper are used in [1], when establishing conditions for linear independence.

In Section 2 we look at the geometry around boxes (a common term for points, line segments, rectangles, 3D-boxes, etc.), box partitions (splitting of a parameter domain into elements) and meshes (the "boundary" that splits the elements, i.e. the parameter domain minus the interior of the elements). Section 3 gives some basic homology concepts. Section 4 is a preparation for the $\mathfrak{R}$-homology before introducing polynomial degrees and multiplicities. The goal is to determine the rank of the homologies. In [3] it is shown that that $H_2(\mathfrak{R}_{m,m'}(\mathcal{T}^o)) \cong R_{m,m'}$ and $H_1(\mathfrak{R}_{m,m'}(\mathcal{T}^o)) = H_0(\mathfrak{R}_{m,m'}(\mathcal{T}^o)) = 0$. These results are lifted to general dimensions: $H_d(\mathfrak{R}^0)$ is the polynomial space, $H_q(\mathfrak{R}^0) = 0$ for $q < d$ for the open homology. For the closed homology $\mathfrak{R}$, the result is the opposite: $H_0(\mathfrak{R})$ is the polynomial space, $H_q(\mathfrak{R}^0) = 0$ for $q > 0$. Therefore we introduce an extra vector space at the end of the chain complex $\mathfrak{R}$ to get the reduced homology $\widetilde{\mathfrak{R}}$ where all homologies are zero, just like what is often done in the q-simplex homology in algebraic topology.

In Section 5 we introduce the homologies $\mathfrak{I}$ and $\mathfrak{S}$, and prove som basic results. The dimension formulas are expressed and proven in Section 6. We explicitely find the homologies for tensor product spaces in Section 7, we have zero-dimensional homologies unless we have far too few knots in some of the directions (actually so few that no nontrivial spline function exists). Though this is a very simple result, it's important as a foundation as all meshes have their origin from tensor product meshes.

Then Section 8 shows that if we take any mesh and extend to a tensor product mesh with zero multiplicities along the added lines in the extension process, we still get the same homologies. This is important, as we can then assume we always work with tensor meshes (with sometimes zero-multiplicities along some meshrectangles). It makes the subdivision process simpler.

We look at subdivision (or multiplicity elevation) at Section 9. The idea is to look at homology connections between meshes before further subdivision, after the division, and the inherited mesh at the division area. In the final work, we might approach this a bit different than what is done now.

The final two sections give some explicit results for dimension 2 and 3. Dimension 2 is very much a repeption of the result in [3].

2 Meshes and box partitions

Many of the definitions in this section follow those of [1], except that we here also allow zero multiplicity along a meshrectangle.

Definition 1. *Given an integer $d \geq 1$. A* **box** *in $\mathbb{R}^d$ is a cartesian product*

$$\beta = J_1 \times \cdots \times J_d \subseteq \mathbb{R}^d \tag{1}$$

where each $J_k = [a_{\beta,k}, b_{\beta,k}]$ is a closed finite interval in $\mathbb{R}^d$. If $a_{\beta,k} = b_{\beta,k}$, then J_k is a point, and is said to be **trivial**. *If $a_{\beta,k} < b_{\beta,k}$, J_k is an interval with non-empty interior, and is said to be* **non-trivial**. *The* **dimension** *of β is the number of non-trivial intervals J_k in (1), it is denoted $\dim \beta$. If $\ell = \dim \beta$, we call β an ℓ-box or (ℓ, d)-box. If $\dim \beta = d$, then β is called an* **element**. *If $\dim \beta = d - 1$, there exists exactly one k such that $J_k = \{a\}$ is trivial. Then β is called a* **meshrectangle**, *a* **k-meshrectangle** *or a* **(k, a)-meshrectangle**.

If $\mathcal{F}$ is a collection of boxes in $\mathbb{R}^d$, we define

$$\mathcal{F}_\ell = \{\beta \in \mathcal{F} : \dim \beta = \ell\} \tag{2}$$

to be the set of all ℓ-dimensional boxes in $\mathcal{F}$ for $\ell = 0, \dots, d$. Sometimes it is convenient to look at boxes with non-trivial components in specific directions. If $\mathcal{F}$ is a collection of boxes in $\mathbb{R}^d$ and for a set $B \subseteq \{1, \dots, d\}$, we define

$$\mathcal{F}_B = \{\beta \in \mathcal{F} : a_{\beta,k} < b_{\beta,k} \Leftrightarrow k \in B \, \forall k\} \tag{3}$$

i.e. $\mathcal{F}_B$ contains all $\beta \in \mathcal{F}$ such that the kth parameter direction of β is non-trivial if and only if $k \in B$. It is a subset of $\mathcal{F}_{|B|}$.

Definition 2. *Given an (ℓ, d)-box $\beta = J_1 \times \cdots \times J_d$, and let $1 \leq k_1 < \ldots < k_\ell \leq d$ be the coordinate directions such that $J_{k_j} = [a_{\beta,k_j}, b_{\beta,k_j}]$ is non-trivial. For any $j = 1, \dots, \ell$ we define the $(\ell - 1, d)$-boxes*

$$\partial_0^j \beta = J_1 \times \ldots \times J_{k_j - 1} \times \{a_{\beta,k_j}\} \times J_{k_j+1} \times \ldots \times J_d \tag{4}$$

$$\partial_1^j \beta = J_1 \times \ldots \times J_{k_j - 1} \times \{b_{\beta,k_j}\} \times J_{k_j+1} \times \ldots \times J_d \tag{5}$$

Each $\partial_i^j \beta$ for $i = 0, 1$ and $j = 1, \dots, \ell$ is called a **face** *of β. The* **boundary** *of β, denoted $\partial\beta$ is defined as*

$$\partial\beta = \bigcup_{\substack{i=0,1 \\ j=1,\dots,\ell}} \partial_i^j \beta \tag{6}$$

If β is a (d, d)-box, then $\partial\beta$ is the topological boundary of β. In general, $\partial\beta$ is the topological boundary of β in the unique linear $(\dim \beta)$-dimensional subspace of $\mathbb{R}^d$ containing β.

For a collection $\mathcal{F}$ of boxes in $\mathbb{R}^d$ and a number $k \in \{1, \dots, d\}$ we define

$$W_k(\mathcal{F}) = \bigcup_{\beta \in \mathcal{F}} \{a_{\beta,k}, b_{\beta,k}\} \tag{7}$$

to be the set of all start and end points in the kth parameter direction of the boxes in $\mathcal{F}$. If $\mathcal{F}$ is finite, $W_k(\mathcal{F})$ is finite, then there exists for every point $\boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$ a real number $\epsilon_{\boldsymbol{q}}(\mathcal{F}) > 0$ (or just $\epsilon_{\boldsymbol{q}}$) such that

$$((q_k - \epsilon_{\boldsymbol{q}}, q_k) \cup (q_k, q_k + \epsilon_{\boldsymbol{q}})) \cap W_k(\mathcal{F}) = \emptyset \ \forall k. \tag{8}$$

Definition 3. *Let $\Omega \subseteq \mathbb{R}^d$ be a d-box in $\mathbb{R}^d$. A finite collection $\mathcal{E}$ of d-boxes in $\mathbb{R}^d$ is said to be a* **box partition** *of Ω if*

1. *$\beta_1^o \cap \beta_2^o = \emptyset$ for any $\beta_1, \beta_2 \in \mathcal{E}$ where $\beta_1 \neq \beta_2$.*
2. *$\bigcup_{\beta \in \mathcal{E}} \beta = \Omega$.*

Given a box partition $\mathcal{E}$ of a box $\Omega \subseteq \mathbb{R}^d$, we want to look at the structure of lower dimensional boxes at the boundary of the partition boxes. For a point $\boldsymbol{q} \in \Omega^o$ and a diagnoal direction $\boldsymbol{\sigma} \in \{-1, 1\}^d$, there is a unique box $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}} \in \mathcal{E}$ such that the ray from $\boldsymbol{q}$ in direction $\boldsymbol{\sigma}$ starts inside $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}}$, i.e. the line segment joining $\boldsymbol{q}$ and $\boldsymbol{q} + \epsilon_{\boldsymbol{q}}(\mathcal{E})\boldsymbol{\sigma}$ is contained in $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}}$. We define the set $\beta_{\boldsymbol{q}}$ to be

$$\beta_{\boldsymbol{q}} = \bigcap_{\boldsymbol{\sigma} \in \{1,1\}^d} X_{\boldsymbol{q}}^{\boldsymbol{\sigma}} \tag{9}$$

It is the intersection of all boxes in $\mathcal{E}$ containing $\boldsymbol{q}$, as any box containing $\boldsymbol{q}$ also must continue in least one diagonal direction from $\boldsymbol{q}$. This construction helps us to find boxes hitting the interior of Ω. If we also want the boundary boxes on Ω, we add surrounding boxes: With $\Omega = J_1 \times \ldots \times J_d$ where $J_k = [a_k, b_k]$ for every k, we define the set Ω^+ by

$$\Omega^+ = \{J_1 \times \ldots \times J_d : J_k \in \{[a_k - 1, a_k], [a_k, b_k], [b_k, b_k + 1]\} \forall k\} \setminus \{\Omega\} \quad (10)$$

The set $\mathcal{E} \cup \Omega^+$ is a box partition of $[a_1 - 1, b_1 + 1] \times \ldots \times [a_d - 1, b_d + 1]$.

Definition 4. *Given a box partition $\mathcal{E}$ of Ω in $\mathbb{R}^d$. We define the sets*

$$\mathcal{F}(\mathcal{E}) := \bigcup_{\boldsymbol{q} \in \Omega} \{\beta_q(\mathcal{E} \cup \Omega^+)\} \quad (11)$$

$$\mathcal{F}^0(\mathcal{E}) := \bigcup_{\boldsymbol{q} \in \Omega^o} \{\beta_q(\mathcal{E})\} \quad (12)$$

$$\mathcal{F}_\ell(\mathcal{E}) := (\mathcal{F}(\mathcal{E}))_\ell = \{\beta \in \mathcal{F}(\mathcal{E}) : \dim \beta = \ell\} \text{ for } \ell = 0, \ldots, d \quad (13)$$

$$\mathcal{F}^0_\ell(\mathcal{E}) := (\mathcal{F}^0(\mathcal{E}))_\ell = \{\beta \in \mathcal{F}^0(\mathcal{E}) : \dim \beta = \ell\} \text{ for } \ell = 0, \ldots, d \quad (14)$$

A box in $\mathcal{F}^0(\mathcal{E})$ is called an **inner** *box of $\mathcal{E}$. A box in $\mathcal{F}(\mathcal{E}) \setminus \mathcal{F}^0(\mathcal{E})$ is called a* **boundary** *box of $\mathcal{E}$. These names fall natural in light of Lemma 2.*

A **boxmesh** *in $\mathbb{R}^d$ is a set $\mathcal{M}$ of meshrectangles given as $\mathcal{M} = \mathcal{F}_{d-1}(\mathcal{E})$ for some box partition $\mathcal{E}$ in $\mathbb{R}^d$.*

Some examples of box partitions and meshrectangles for dimension 1, 2 and 3 are shown in Figure 1.

Lemma 1. *Given a box partition $\mathcal{E}$ on a d-box Ω and a point $\boldsymbol{q} \in \Omega^o$.*

1. *For any point $\boldsymbol{r} \in \beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}}$, we have $\beta_{\boldsymbol{r}} = \beta_{\boldsymbol{q}}$*
2. *$\boldsymbol{q} \in \beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}}$*

Proof.

1. By coordinate permutations, we way assume

$$\beta_{\boldsymbol{q}} = J_1 \times \ldots \times J_\ell \times \{q_{\ell+1}\} \times \ldots \times \{q_d\} \quad (15)$$

where each J_k is nontrivial. Given $\boldsymbol{r} \in \beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}}$, then

$$\boldsymbol{r} = (r_1, \ldots, r_\ell, q_{\ell+1}, \ldots, q_d) \quad (16)$$

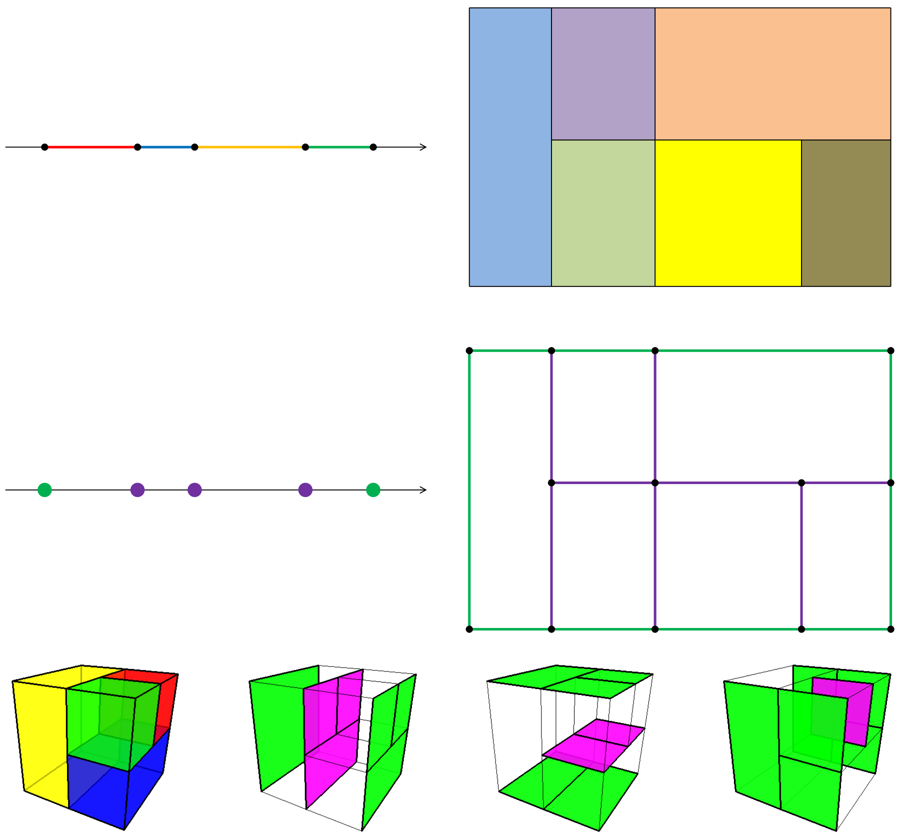


Figure 1: **Box partitions and meshrectangles**. To the upper left, a box partition in $\mathbb{R}$, with a different color for each of the four elements. The elements are line segments. Below are the meshrectangles, with different colors for the three inner and two boundary meshrectangles. Meshrectangles in $\mathbb{R}$ are points. To the upper right, a box partition in $\mathbb{R}^2$ and the meshrectangles below. The elements are 2-dimensional rectangles, the meshrectangles are line segments. At the bottom line, a box partition in $\mathbb{R}^3$ to the left where the elements are 3-dimensional. Then three figures of inner and boundary meshrectangles, one for each parameter direction. The meshrectangles are 2-dimensional

where $r_k \in J_k^o$ for each k. Given a direction $\boldsymbol{\sigma} \in \{-1, 1\}^d$, we then have the boxes

$$X_{\boldsymbol{q}}^{\boldsymbol{\sigma}} = K_1 \times \ldots \times K_d \tag{17}$$

$$X_{\boldsymbol{r}}^{\boldsymbol{\sigma}} = L_1 \times \ldots \times L_d \tag{18}$$

If $k > \ell$, then both q_k and $q_k + \sigma_k \min(\epsilon_{\boldsymbol{q}}, \epsilon_{\boldsymbol{r}})$ lie in both K_k and L_k, and so $K_k^o \cap L_k^o \neq \emptyset$. If $k \leq \ell$, then $r_k \in J_k^o \subseteq K_k^o$ and $r_k \in L_k$, hence again $K_k^o \cap L_k^o \neq \emptyset$. This means that $(X_{\boldsymbol{q}}^{\boldsymbol{\sigma}})^o \cap (X_{\boldsymbol{r}}^{\boldsymbol{\sigma}})^o \neq \emptyset$, and because different boxes in $\mathcal{E}$ have disjoint interior, we have $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}} = X_{\boldsymbol{r}}^{\boldsymbol{\sigma}}$. Intersecting this over all $\boldsymbol{\sigma}$ yields $\beta_{\boldsymbol{r}} = \beta_{\boldsymbol{q}}$.

2. Suppose $\boldsymbol{q} \in \partial\beta_{\boldsymbol{q}}$. We can transform to $\beta_{\boldsymbol{q}} = [0, 1]^\ell \times \{0\}^{d-\ell}$ and $q_1 = 0$. Because the first component of $\beta_{\boldsymbol{q}}$ starts at 0, then the first component of $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}}$ must start at 0 for some $\boldsymbol{\sigma}$ such that $\sigma_1 = 1$. If we put $\boldsymbol{\sigma}' = (-1, \sigma_2 \ldots, \sigma_d)$, we must have $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}} \neq X_{\boldsymbol{q}}^{\boldsymbol{\sigma}'}$. Then the first component of $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}'}$ must end at 0 because the kth component of $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}}$ and $X_{\boldsymbol{q}}^{\boldsymbol{\sigma}'}$ have the interval between q_k and $q_k + \epsilon_{\boldsymbol{q}}\sigma_k$ in common for $k \geq 2$. This contradicts that the first component of $\beta_{\boldsymbol{q}}$ is $[0, 1]$. Therefore $\boldsymbol{q} \in \beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}}$.

□

Lemma 2. *Given a box partition $\mathcal{E}$ on a d-box Ω.*

1. *Every box in $\mathcal{F}(\mathcal{E})$ is contained in Ω.*

2. *$\mathcal{F}^0(\mathcal{E}) \subseteq \mathcal{F}(\mathcal{E})$. A box in $\mathcal{F}(\mathcal{E})$ is an inner box if it intersects nonempty with Ω^o, and a boundary box if it lies on $\partial\Omega$.*

3. *$\mathcal{F}_d^0(\mathcal{E}) = \mathcal{F}_d(\mathcal{E}) = \mathcal{E}$.*

Proof.

1. Given $\boldsymbol{q} \in \Omega$, then at least one box $\beta \in \mathcal{E}$ contains $\boldsymbol{q}$, then $\beta_{\boldsymbol{q}} \subseteq \beta \subseteq \Omega$

2. Given an inner box $\beta_{\boldsymbol{q}}(\mathcal{E})$ where $\boldsymbol{q} \in \Omega^o$. As $\boldsymbol{q}$ is not contained in any box in Ω^+, we must have $\beta_{\boldsymbol{q}}(\mathcal{E}) = \beta_{\boldsymbol{q}}(\mathcal{E} \cup \Omega^+)$. Also, since $\boldsymbol{q} \in \Omega^o \cap \beta_{\boldsymbol{q}}$ we see that $\mathcal{F}^0(\mathcal{E}) \subseteq \mathcal{F}(\mathcal{E})$ and any inner box in $\mathcal{F}(\mathcal{E})$ intersects nonempty with Ω^o. If $\beta_{\boldsymbol{q}}(\mathcal{E} \cap \Omega^+)$ is not in $\mathcal{F}^0(\mathcal{E})$, we must have $\boldsymbol{q} \in \partial\Omega$, then $\boldsymbol{q}$ lies in at lest one box in Ω^+, therefore $\beta_{\boldsymbol{q}}(\mathcal{E} \cap \Omega^+)$ is contained in $\partial\Omega$.

3. $\mathcal{F}_d^0(\mathcal{E}) \subseteq \mathcal{F}_d(\mathcal{E})$ is obvious from the above. Given $\beta_{\boldsymbol{q}}(\mathcal{E} \cup \Omega^+) \in \mathcal{F}_d(\mathcal{E})$, it lies in at least one box in $\beta \in \mathcal{E}$. Two different boxes in a box partition have disjoint interior and therefore intersect down to a box of dimension $< d$. Therefore, since $\dim \beta_{\boldsymbol{q}} = d$, β is the only box in $\mathcal{E} \cup \Omega^+$ containing $\boldsymbol{q}$. Hence $\beta_{\boldsymbol{q}} = \beta$ and so $\mathcal{F}_d(\mathcal{E}) \subseteq \mathcal{E}$. Finally, for a box $\beta \in \mathcal{E}$, pick a point $\boldsymbol{q} \in \beta^o$. No other box in $\mathcal{E}$ can contain $\boldsymbol{q}$, and so $\beta_{\boldsymbol{q}} = \beta$ giving $\mathcal{E} \subseteq \mathcal{F}_d^0(\mathcal{E})$. Alltogether we have $\mathcal{F}_d^0(\mathcal{E}) \subseteq \mathcal{F}_d(\mathcal{E}) \subseteq \mathcal{E} \subseteq \mathcal{F}_d^0(\mathcal{E})$ givng the result.

□

We introduce the notion of homology-suitable box sets. The reason for this naming will become clear when we later show how we can impose a homology structure based on a homology suitable set.

Definition 5. *A finite collection $\mathcal{F}$ of boxes in $\mathbb{R}^d$ of various dimensions is* **homology-suitable** *if*

1. *For every $\ell \geq 1$ and $\beta \in \mathcal{F}_\ell$ there is a sequence $\gamma_1, \ldots, \gamma_n \in \mathcal{F}_{\ell-1}$ such that $\partial\beta = \gamma_1 \cup \ldots \cup \gamma_n$*

2. *For two different $\beta_1, \beta_2 \in \mathcal{F}$ such that $\dim \beta_1 \geq \dim \beta_2$, we have $\beta_2 \cap (\beta_1 \setminus \partial\beta_1) = \emptyset$.*

3. *$\mathcal{F}_d$ is a box partition of a d-box.*

The d-box being partitioned by $\mathcal{F}_d$ is denoted $\Omega(\mathcal{F})$

Lemma 3. *Given a box partition $\mathcal{E}$ of Ω in $\mathbb{R}^d$. Then $\mathcal{F}(\mathcal{E})$ is homology suitable.*

Proof.

1. By linear transformations and coordinate permuations, it sufficies to show that the boundary of $\beta = [0,1]^\ell \times \{0\}^{d-\ell}$ is a union of boxes in $\mathcal{F}_{\ell-1}(\mathcal{E})$. Define the set

$$X = \{0\} \times ([0,1] \setminus W_2) \times \ldots \times ([0,1] \setminus W_\ell) \times \{0\}^{d-\ell} \tag{19}$$

Given $\boldsymbol{q} \in X$. Then $\beta_{\boldsymbol{q}} \subseteq \beta$ because $\boldsymbol{q} \in \beta$, therefore

$$\beta_{\boldsymbol{q}} = J_1 \times J_2 \times \ldots \times J_\ell \times \{0\}^{d-\ell} \tag{20}$$

where $J_k \subseteq [0,1]$ when $k \leq \ell$. For $k = 2, \ldots, \ell$, J_k is nontrivial because $q_k \notin W_k$, so we have $\dim \beta_{\boldsymbol{q}} \geq \ell - 1$. Assume J_1 is not trivial. Since $\boldsymbol{q} \in (\beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}})$ by Lemma 1, we must have $0 \in J_1^o$. Then there is a $r_1 \in J_1^0 \cap (0,1)$. But then the point $\boldsymbol{r} = (r_1, q_2, \ldots, q_d)$ lies both in $\beta_{\boldsymbol{q}} \setminus \partial\beta_{\boldsymbol{q}}$ and $\beta \setminus \partial\beta$, then (by Lemma 1 again) $\beta_{\boldsymbol{q}} = \beta_{\boldsymbol{r}} = \beta$, which is impossible as $\boldsymbol{q} \in \partial\beta$. Therefore $J_1 = \{0\}$ and $\dim \beta_{\boldsymbol{q}} = \ell - 1$. We have inclusions

$$X \subseteq \bigcup_{\boldsymbol{q} \in X} \beta_{\boldsymbol{q}} \subseteq \partial_1^0 \beta \tag{21}$$

The union is finite (beacuse $\mathcal{F}(\mathcal{E})$ is finite) and of closed sets, and X is dense in $\partial_1^0\beta$, therefore $\cup_{\boldsymbol{q}\in X}\beta_{\boldsymbol{q}} = \partial_1^0\beta$. This holds for any face of β, and so $\partial\beta$ is a finite union of boxes in $\mathcal{F}_{\ell-1}(\mathcal{E})$.

2. Given two different $\beta_1, \beta_2 \in \mathcal{F}$ such that $\dim \beta_1 \geq \dim \beta_2$, and suppose there exists a point $\boldsymbol{q} \in \beta_2 \cap (\beta_1 \setminus \partial\beta_1)$. Then $\beta_1 = \beta_{\boldsymbol{q}}$ by Lemma 1. Also $\boldsymbol{q} \in \beta_2$ gives $\beta_1 = \beta_{\boldsymbol{q}} \subseteq \beta_2$, hence $\beta_1 \setminus \partial\beta_1 \subseteq \beta_2 \setminus \partial\beta_2$ because $\dim \beta_1 \geq \dim \beta_2$. Then for any point $\boldsymbol{r} \in \beta_1 \setminus \partial\beta_1$ we get $\beta_1 = \beta_{\boldsymbol{r}} = \beta_2$. But this is impossible because $\beta_1 \neq \beta_2$. Therefore no such $\boldsymbol{q}$ exists, hence $\beta_2 \cap (\beta_1 \setminus \partial\beta_1) = \emptyset$.

3. $\mathcal{F}_d(\mathcal{E}) = \mathcal{E}$ is a box partition of Ω.

□

Lemma 4. *Let $\mathcal{F}$ be a homology-suitable collection of boxes in $\mathbb{R}^d$. Then*

$$\bigcup_{\beta \in \mathcal{F}_{d-1}} \beta = \Omega(\mathcal{F}) \setminus \bigcup_{\gamma \in \mathcal{F}_d} \gamma^o \tag{22}$$

Proof. Clearly, the righthand side of (22) is the same as $\bigcup_{\gamma \in \mathcal{F}_d} \partial\gamma$. The result then follows easily from the definition of homology-suitable. □

Let $\mathcal{E}$ be a box partition in $\mathbb{R}^d$, and $\mathcal{M} = \mathcal{F}_{d-1}(\mathcal{E})$ be the boxmesh. The box Ω being partitioned by $\mathcal{E}$ is given as the smallest box containing every meshrectangle in $\mathcal{M}$. From Lemma 4, we see that each topologcal connection component of $\Omega \setminus (\bigcup_{\beta \in \mathcal{M}} \beta)$ is the interior of a box $\mathcal{E}$. This way we can reconstruct $\mathcal{E}$ from only knowing the boxmesh $\mathcal{M}$. If $\mathcal{M}$ is a boxmesh, we use $\mathcal{E}(\mathcal{M})$ to denote the unique box partition such that $\mathcal{M} = \mathcal{M}(\mathcal{E}(\mathcal{M}))$.

We also define the sets

$$\mathcal{F}(\mathcal{M}) := \mathcal{F}(\mathcal{E}(\mathcal{M})) \tag{23}$$

$$\mathcal{F}_\ell(\mathcal{M}) := \mathcal{F}_\ell(\mathcal{E}(\mathcal{M})) \text{ for any } \ell = 0, \dots, d \tag{24}$$

$$\beta_{\boldsymbol{q}}(\mathcal{M}) := \beta_{\boldsymbol{q}}(\mathcal{E}(\mathcal{M})) \text{ for any } \boldsymbol{q} \in \Omega \tag{25}$$

Definition 6. *Let $\mathcal{M}$ be a boxmesh in $\mathbb{R}^d$, and given a box $\beta \in \mathcal{F}(\mathcal{M})$. Define the sets*

$$\mathcal{D}_\beta(\mathcal{M}) = \{\gamma \in \mathcal{M} : \beta \subseteq \gamma\} \tag{26}$$

$$\mathcal{D}^k_\beta(\mathcal{M}) = \{\gamma \in \mathcal{D}_\beta(\mathcal{M}) : \gamma \textit{ is a } k\textit{-meshrectangle}\} \tag{27}$$

for $1 \leq k \leq d$.

Clearly $\mathcal{D}_\beta(\mathcal{M}) = \bigcup_{k=1}^d \mathcal{D}^k_\beta(\mathcal{M})$. If the kth component of β is non-trivial, then $\mathcal{D}^k_\beta(\mathcal{M}) = \emptyset$. In particular if $\dim \beta = d$ then $\mathcal{D}_\beta(\mathcal{M}) = \emptyset$. If $\dim \beta = d - 1$ then $\mathcal{D}_\beta(\mathcal{M}) = \{\beta\}$.

Definition 7. *Given a sequence $\mathcal{M}_1, \dots, \mathcal{M}_n$ of boxmeshes, where $\mathcal{M}_k$ is a boxmesh in $\mathbb{R}^{d_k}$ for $k = 1, \dots, n$. We define $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ to be the boxmesh given by*

$$\mathcal{E}(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n) = \{\beta_1 \times \cdots \times \beta_n : \beta_k \in \mathcal{E}(\mathcal{M}_k) \forall k\}, \tag{28}$$

this is a mesh in $\mathbb{R}^{d_1 + \cdots + d_n}$.

Definition 8. *Given a sequence $\boldsymbol{a} = (a_1, \dots, a_n)$ of length ≥ 2 where $a_i < a_{i+1}$ for $1 \leq i \leq n-1$. We define $\mathcal{M}(\boldsymbol{a})$ to be the mesh in $\mathbb{R}$ where the points $\{a_i\}$ are the meshrectangles. The set $\mathcal{E}(\mathcal{M}(\boldsymbol{a}))$ is the set of all intervals $[a_i, a_{i+1}]$ in $\mathbb{R}$, it is a box partition of $[a_1, a_n]$.*

Given a sequence $\boldsymbol{a}_1, \dots, \boldsymbol{a}_d$ where each $\boldsymbol{a}_k = (a_{k,1}, \dots, a_{k_{n_k}})$ is a finite increasing sequence. The **tensor product mesh** *$\mathcal{M}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_d)$ is the d-dimensional mesh*

$$\mathcal{M}(\boldsymbol{a}_1, \dots, \boldsymbol{a}_d) = \mathcal{M}(\boldsymbol{a}_1) \times \ldots \times \mathcal{M}(\boldsymbol{a}_d) \tag{29}$$

Definition 9. *If $\mathcal{M}$ is a mesh in $\mathbb{R}^d$, we define the* **tensor expansion** *of $\mathcal{M}$ to be*

$$\mathcal{M}^T = \mathcal{M}(W_1(\mathcal{E}(\mathcal{M})), \dots, W_d(\mathcal{E}(\mathcal{M}))), \tag{30}$$

it is the "smallest" tensor product mesh containing $\mathcal{M}$.

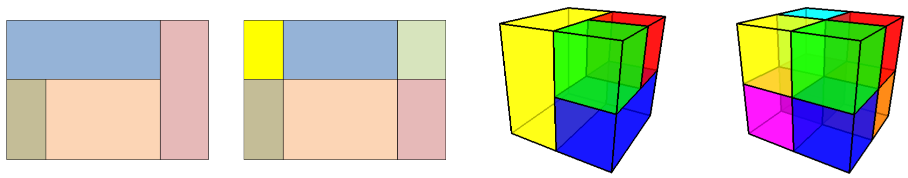


Figure 2: To the left a box partition $\mathcal{M}$ in $\mathbb{R}^2$, and the tensor expansion $\mathcal{M}^T$ of $\mathcal{M}$. To the right a box partition and its tensor expansion in $\mathbb{R}^3$.

Two examples of a box partition and its tensor expansion is shown in Figure 2.

We are going to look at the spline functions on a boxmesh. First we need the set of polynomial functions. We define $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ to be the polynomial ring in d variables over $\mathbb{R}$. For a sequence $\boldsymbol{p} = (p_1, \ldots, p_d)$ of positive integers, we define $\Pi_{\boldsymbol{p}} \subset \Pi^d$ to be the vector space spanned by the monomials $x_1^{i_1} \ldots x_d^{i_d}$ in Π^d such that $0 \leq i_k \leq p_k$ for all $k = 1, \ldots, d$.

For a box partition $\mathcal{E}$ of $\Omega = [a_1, b_1] \times \cdots \times [a_d, b_d]$ and a box $\beta \in \mathcal{E}$ on the form (1), we let $\tilde{\beta}$ be the set $J_1 \times \ldots \times J_d \subseteq \mathbb{R}^d$ where $J_k = [a_{\beta,k}, b_{\beta,k})$ if $b_{\beta,k} < b_k$ and $J_k = [a_{\beta,k}, b_{\beta,k}]$ if $b_{\beta,k} = b_k$. The only reason for this mix of closed and half-open intervals is to get a valid definition of spline spaces below in case of C^{-1}-continuities.

Definition 10. *A* **spline mesh** *is a tripple $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ where $\mathcal{M}$ is a boxmesh from a box partition of a box $\Omega \subseteq \mathbb{R}^d$, $\boldsymbol{p} = (p_1, \ldots, p_d)$ is a sequence of positive numbers and $\mu : \mathcal{M} \to \mathbb{Z}$ is a map such that $0 \leq \mu(\beta) \leq p_k + 1$ for every k-meshrectangle $\beta \in \mathcal{M}$.*

The **spline space** *defined by $\mathcal{N}$, denoted $\mathbb{S}(\mathcal{N})$ is the set of all functions $f : \mathbb{R}^d \to \mathbb{R}$ such that*

1. *f is zero outside Ω.*
2. *For each box $\gamma \in \mathcal{E}(\mathcal{M})$, the restriction of f to $\tilde{\gamma}$ is a d-variate polynomial function in $\Pi_{\boldsymbol{p}}$.*
3. *For each k-meshrectangle $\beta \in \mathcal{M}$, f has $C^{p_k - \mu(\beta)}$-continuity along β.*

Point 3. above could also be formulated as follows: For every (k, a)-meshrectangle $\beta \in \mathcal{M}$, and every pair γ_1, γ_2 of sets containing β in $\mathbb{R}^d$ such that each γ_i is either a box in $\mathcal{E}(\mathcal{M})$ or the set $\mathbb{R}^d \setminus \Omega(\mathcal{M})^o$, let f_1 and f_2 be the polynomial expressions for f on γ_1 and γ_2 respectively, then $f_1 - f_2 = F \cdot (x_k - a)^{p_k - \mu(\beta)+1}$ for some polynomial function F.

The purpose of this paper is to give an expression for the dimension of the spline space $\mathbb{S}(\mathcal{N})$.

Definition 11. *A spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ is* **open** *if $\mu(\beta) = p_k + 1$ for every boundary k-meshrectangle β in $\mathcal{M}$.*

Definition 12. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ and a box $\beta \in \mathcal{F}(\mathcal{M})$. We define*

$$\mu_k(\beta) = \max(\{0\}, \{\mu(\gamma) : \gamma \in \mathcal{D}^k_\beta(\mathcal{M})\}) \tag{31}$$

for $1 \leq k \leq d$.

Notice that if the kth component of β is non-trivial, then $\mu_k(\beta) = 0$. If β is a k-meshrectangle, then $\mu_k(\beta) = \mu(\beta)$.

Definition 13. *Given an increasing sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ of real numbers, a positive integer p and a sequence $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ such that $0 \leq \mu_i \leq p + 1$ for all i. Then we define the* **univariate spline mesh** *$\mathcal{N}(\boldsymbol{a}, \boldsymbol{\mu}, p)$ by*

$$\mathcal{N}(\boldsymbol{a}, \boldsymbol{\mu}, p) = (\mathcal{M}(\boldsymbol{a}), \{a_i\} \mapsto \mu_i, (p)) \tag{32}$$

If $\mathcal{N}_1, \ldots, \mathcal{N}_d$ are univariate spline meshes such that $\mathcal{N}_k = \mathcal{N}(\boldsymbol{a}_k, \boldsymbol{\mu}_k, p_k)$ for some sequences $\boldsymbol{a}_k = (a_{k,1}, \ldots, a_{k,n_k})$ and $\boldsymbol{\mu}_k = (\mu_{k,1}, \ldots, \mu_{k,n_k})$, we define the **tensor product spline mesh**

$$\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_d \tag{33}$$

to be the spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ where

- $\mathcal{M} = \mathcal{M}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$
- $\boldsymbol{p} = (p_1, \ldots, p_d)$
- *For every $(k, a_{k,i})$-meshrectangle $\beta \in \mathcal{M}$, $\mu(\beta) = \mu_{k,i}$.*

Notice that $\mathbb{S}(\mathcal{N}(\boldsymbol{a}, \boldsymbol{\mu}, p))$ is the space of univariate p-degree spline functions defined by the knot vector where each a_i occurs μ_i times, while $\mathbb{S}(\mathcal{N}_1 \times \cdots \times \mathcal{N}_d)$ is the spline space of d-variate tensor product spline functions defined by the given degrees, knots and multiplicities in each parameter direction.

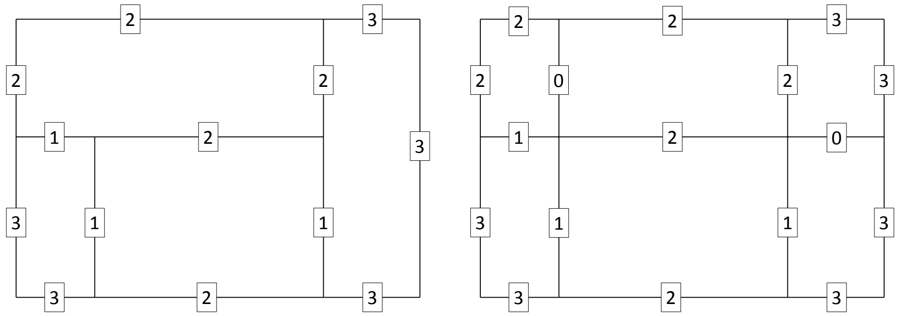


Figure 3: To the left, a spline mesh $\mathcal{N}$ in $\mathbb{R}^2$ with the multiplicities $\mu(\beta)$ written on each meshrectangle. To the right, the tensor expansion $\mathcal{N}^T$ of $\mathcal{N}$.

Definition 14. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh. We define the* **tensor expansion** *of $\mathcal{N}$ to be the spline mesh*

$$\mathcal{N}^T = (\mathcal{M}^T, \mu^T, \boldsymbol{p}) \tag{34}$$

where the mapping $\mu^T : \mathcal{M}^T \to \mathbb{Z}$ is defined as

$$\mu^T(\beta) = \begin{cases} \mu(\gamma) & \text{if } \beta \subseteq \gamma \text{ for some } \gamma \in \mathcal{M} \\ 0 & \text{if } \beta \nsubseteq \gamma \text{ for all } \gamma \in \mathcal{M} \end{cases} \tag{35}$$

Note that μ^T is well-defined because there is maximum one $\gamma \in \mathcal{M}$ such that $\beta \subseteq \gamma$, another $\gamma' \in \mathcal{M}$ will intersect with γ down to a box of dimension $\leq d - 2$ by Definition 5, and can therefore not contain β.

See Figure 3 for an example of a tensor expansion in $\mathbb{R}^2$.

3 Homology

We recall some definitions and properties about chain complexes and homology. For details, see [2]. All vector spaces used will be over the real numbers.

Definition 15. *Given a sequence of vector spaces V_i, $i = n, \ldots, m$ for some $n \leq m$, and linear maps $\delta_q : V_q \to V_{q-1}$. The complex*

$$\mathfrak{V} : V_m \xrightarrow{\delta_m} V_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_{n+1}} V_n \tag{36}$$

is a **chain complex** *if* $\operatorname{im}\delta_{q+1} \subseteq \ker\delta_q$ *for* $n+1 \leq q \leq m-1$*. For a chain complex on the form* (36)*, we define the* q**th homology of** $\mathfrak{V}$ *to be the vector space* $H_q(\mathfrak{V}) = \ker\delta_q / \operatorname{im}\delta_{q+1}$*. A chain complex is an* **exact sequence** *if* $\operatorname{im}\delta_{q+1} = \ker\delta_q$ *(or equivalent:* $H_q(\mathfrak{V}) = 0$*) for all* q*.*

When we work with chain complexes and exact sequences, the first and last vector space in the complex will always be zero.

Short exact sequences of chain complexes give rise to the important long exact sequence of homologies

Lemma 5. *Given a commutative diagram*

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & & & 0 & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \mathfrak{A}:0 & \to & A_m & \to & A_{m-1} & \to & \cdots & \to & A_n & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \mathfrak{B}:0 & \to & B_m & \to & B_{m-1} & \to & \cdots & \to & B_n & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \mathfrak{C}:0 & \to & C_m & \to & C_{m-1} & \to & \cdots & \to & C_n & \to & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ & & 0 & & 0 & & & & 0 & & \end{array} \tag{37}$$

of vector spaces such that the rows $\mathfrak{A}, \mathfrak{B}$ *and* $\mathfrak{C}$ *are chain complexes, and all the columns are exact sequences. Then there is an exact sequence*

$$0 \to H_m(\mathfrak{A}) \to H_m(\mathfrak{B}) \to H_m(\mathfrak{C}) \to H_{m-1}(\mathfrak{A}) \to \cdots \to H_n(\mathfrak{C}) \to 0 \tag{38}$$

Proof. See [2] □

We will also use the fact that alternating sums of vector space dimensions equals the alternating sum of homologies in a chain complex.

Lemma 6. *Given a chain complex*

$$\mathfrak{A} : 0 \xrightarrow{\delta_{m+1}} A_m \xrightarrow{\delta_m} A_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_{n+1}} A_n \xrightarrow{\delta_n} 0 \tag{39}$$

of finitedimensional vector spaces. Then

$$\sum_{i=n}^{m} (-1)^i \dim A_i = \sum_{i=n}^{m} (-1)^i \dim H_i(\mathfrak{A}) \tag{40}$$

Proof. Notice that $\dim \operatorname{im} \delta_{m+1} = \dim \operatorname{im} \delta_n = 0$, and that for every i, we have $\dim \ker \delta_i + \dim \operatorname{im} \delta_i = \dim A_i$. Then

$$\begin{aligned}\sum_{i=n}^{m} (-1)^i \dim A_i &= \sum_{i=n}^{m} (-1)^i (\dim \ker \delta_i + \dim \operatorname{im} \delta_i) \\ &= \sum_{i=n}^{m} (-1)^i (\dim \ker \delta_i - \dim \operatorname{im} \delta_{i+1}) \\ &= \sum_{i=n}^{m} (-1)^i \dim H_i(\mathfrak{A})\end{aligned} \tag{41}$$

□

We will need the tensor product of chain complexes.

Definition 16. *Given chain complexes $\mathfrak{A} = (A_i)$ and $\mathfrak{B} = (B_i)$. We can then create the* **tensor product chain complex** $\mathfrak{A} \otimes \mathfrak{B}$*, where the qth module is $(\mathfrak{A} \otimes \mathfrak{B})_q = \oplus_i (A_i \otimes B_{q-i})$, with boundary maps $\delta_q : (\mathfrak{A} \otimes \mathfrak{B})_q \to (\mathfrak{A} \otimes \mathfrak{B})_{q-1}$ given by $\delta_q(a \otimes b) = \delta_i a \otimes b + (-1)^i a \otimes \delta_{q-i} b$ where $a \in A_i$ and $b \in B_{q-i}$.*

Lemma 7. *Given chain complexes $\mathfrak{A}$ and $\mathfrak{B}$, then*

$$H_q(\mathfrak{A} \otimes \mathfrak{B}) \cong \bigoplus_i (H_i(\mathfrak{A}) \otimes H_{q-i}(\mathfrak{B})) \tag{42}$$

Proof. If R is a principle ideal domain, the algebraic version of the Künneth formula (see [2]) gives an exact sequence

$$\begin{aligned}0 \to \oplus_i (H_i(\mathfrak{A}) \otimes_R H_{q-i}(\mathfrak{B})) \to H_q(\mathfrak{A} \otimes_R \mathfrak{B}) \\ \to \oplus_i \operatorname{Tor}_R(H_i(\mathfrak{A}), H_{q-i-1}(\mathfrak{B})) \to 0\end{aligned} \tag{43}$$

where Tor is the Tor functor. If R is a field (as in our case), the Tor functor is zero, giving the result. □

4 Homologies on box collections

Let $\mathcal{F}$ be a homology-suitable box collection over $\mathbb{R}^d$. For $q = 0, \dots, d$ we let $R_q(\mathcal{F})$ and $R_q^0(\mathcal{F})$ be the free $\mathbb{R}$-modules over all q-dimensional boxes in $\mathcal{F}$

and $\mathcal{F}^0$ respectively, i.e.

$$R_q(\mathcal{F}) = \bigoplus_{\beta\in\mathcal{F}_q} [\beta]\mathbb{R} \tag{44}$$

$$R_q^0(\mathcal{F}) = \bigoplus_{\beta\in\mathcal{F}_q^0} [\beta]\mathbb{R} \tag{45}$$

If $q > 0$, we can also define linear maps $\delta_q : R_q(\mathcal{F}) \to R_{q-1}(\mathcal{F})$ and $\delta_q^0 : R_q^0(\mathcal{F}) \to R_{q-1}^0(\mathcal{F})$ by

$$\delta_q([\beta]) = \sum_{i=1}^{q}\sum_{j=0}^{1}(-1)^{i+j} \sum_{\substack{\gamma\in\mathcal{F}_{q-1}\\ \gamma\subseteq\partial_i^j\beta}} [\gamma] \tag{46}$$

$$\delta_q^0([\beta]) = \sum_{i=1}^{q}\sum_{j=0}^{1}(-1)^{i+j} \sum_{\substack{\gamma\in\mathcal{F}_{q-1}^0\\ \gamma\subseteq\partial_i^j\beta}} [\gamma] \tag{47}$$

for any $\beta \in \mathcal{F}_q$ and $\beta \in \mathcal{F}_q^0$ respectively.

Lemma 8. *Let $\mathcal{F}$ be a homology-suitable collection of boxes in $\mathbb{R}^d$. Given $\ell \geq 2$ and $\alpha \in \mathcal{F}_\ell$ and $\gamma \in \mathcal{F}_{\ell-2}$ such that $\gamma \subseteq \partial\alpha$. Then there are exactly two $(\ell-1)$-boxes $\beta_1, \beta_2 \in \mathcal{F}_{\ell-1}$ with the property $\beta_i \subseteq \partial\alpha$ and $\gamma \subseteq \partial\beta_i$, and we have exactly one of the following two cases. Either*

1. *$\beta_1, \beta_2 \subseteq \partial_i^j\alpha$ for some i, j, and $\gamma \subseteq \partial_{i'}^0\beta_1$ and $\gamma \subseteq \partial_{i'}^1\beta_2$ for some i' or*
2. *$\beta_1 \subseteq \partial_{i_1}^{j_1}\alpha$ and $\beta_2 \subseteq \partial_{i_2}^{j_2}\alpha$ for some i_1, j_1, i_2, j_2 where $i_i < i_2$, and $\gamma \subseteq \partial_{i_2-1}^{j_2}\beta_1$ and $\gamma \subseteq \partial_{i_1}^{j_1}\beta_2$.*

Proof. After some permutation and linear transformations on the paramter directions, we may assume

$$\alpha = [0,1]^\ell \times \{0\}^{d-\ell} \tag{48}$$

$$\gamma = \{0\} \times \{a\} \times J_3 \times \ldots \times J_\ell \times \{0\}^{d-\ell} \tag{49}$$

where $J_k \subseteq [0,1]$ is nontrivial for $k = 3, \ldots, \ell$, and $0 \leq a \leq \frac{1}{2}$. Then $\gamma \subseteq \partial_1^0\alpha$.

Let $\boldsymbol{q} = (0, a, q_3, q_4, \ldots, q_\ell, 0, \ldots, 0) \in \gamma$ where $q_k \in J_k$, $q_k \notin W_k(\mathcal{F})$, see (7). From Definition 5, there is a $\beta_1 \in \mathcal{F}_{\ell-1}$ such that $\boldsymbol{q} \in \beta_1 \subseteq \partial_1^0\alpha$ and such that

$$\beta_1 \supset \{0\} \times (a, a+\epsilon_{\boldsymbol{q}}) \times (q_3-\epsilon_{\boldsymbol{q}}, q_3+\epsilon_{\boldsymbol{q}}) \times \ldots \times (q_\ell-\epsilon_{\boldsymbol{q}}, q_\ell+\epsilon_{\boldsymbol{q}}) \times \{0\}^{d-\ell} \tag{50}$$

for $\epsilon_{\boldsymbol{q}} = \epsilon_{\boldsymbol{q}}(\mathcal{F})$, see (8). If the second component of β_1 also contains $(a - \epsilon_{\boldsymbol{q}}, a)$, then $\boldsymbol{q} \in \beta \backslash \partial\beta$, and so $\gamma \cap (\beta \backslash \partial\beta) \neq \emptyset$, contradicting Definition 5. Therefore, the second component of β_1 starts at a.

Definition 5 gives us a $\gamma' \in \mathcal{F}_{\ell-2}$, $\gamma' \subseteq \partial_1^0 \beta_1$ such that

$$\gamma' \supset \{0\} \times \{a\} \times (q_3 - \epsilon_{\boldsymbol{q}}, q_3 + \epsilon_{\boldsymbol{q}}) \times \ldots \times (q_\ell - \epsilon_{\boldsymbol{q}}, q_\ell + \epsilon_{\boldsymbol{q}}) \times \{0\}^{d-\ell} \quad (51)$$

But then $\dim(\gamma \cap \gamma') = \ell - 2 = \dim(\gamma)$, then Definition 5 implies $\gamma = \gamma'$, and so $\gamma \subseteq \partial_1^0 \beta_1$. We now have two cases.

1. $a > 0$: We can use the same argument as above to get $\beta_2 \in \mathcal{F}_{\ell-1}$ such that $\beta_2 \subseteq \partial_1^0 \alpha$ and $\gamma \subseteq \partial_1^1 \beta_2$, where the second component of β_2 ends at a. Clearly $\beta_1 \neq \beta_2$. Suppose there is a $\beta \in \mathcal{F}_{\ell-1}$ such that $\beta \subseteq \partial\alpha$ and $\gamma \subseteq \partial\beta$. Then $\beta \subseteq \partial_1^0 \alpha$, and we have

$$\beta \supset \{0\} \times U \times (q_3 - \epsilon_{\boldsymbol{q}}, q_3 + \epsilon_{\boldsymbol{q}}) \times \ldots \times (q_\ell - \epsilon_{\boldsymbol{q}}, q_\ell + \epsilon_{\boldsymbol{q}}) \times \{0\}^{d-\ell} \quad (52)$$

   where either $U \supset (a - \epsilon_{\boldsymbol{q}}, a)$ or $U \supset (a, a + \epsilon_{\boldsymbol{q}})$, giving $\dim(\beta \cap \beta_i) = \ell - 1$ for $i = 1$ or 2, hence $\beta = \beta_1$ or $\beta = \beta_2$. Finally, if we reverse the permutations and linear transformations to the original α, we have $\beta_1, \beta_2 \subseteq \partial_i^j \alpha$ and $\gamma \subseteq \partial_{i'}^0 \beta_1 \cap \partial_{i'}^1 \beta_2$.

2. $a = 0$: This time $\gamma \subseteq \partial_1^0 \alpha \cap \partial_2^0 \alpha$. Same argument as before gives $\gamma \subseteq \partial\beta_2$ for some $\beta_2 \subseteq \partial_2^0 \alpha$, and that these are the only $(\ell - 1)$-boxes $\beta \subseteq \partial\alpha$ such that $\gamma \subseteq \partial\beta$. Moving back to the original α, we have $\beta_1 \subseteq \partial_{i_1}^{j_1} \alpha$ and $\beta_2 \subseteq \partial_{i_2}^{j_2} \alpha$ for $i_1 < i_2$ (swap β_1 and β_2 if necessary). The increasing sequence of parameter directions where β_1 has a nontrivial interval is the same as for α except that i_1 is removed. As $i_1 < i_2$, the position of i_2 in the sequence has decreased by one, therefore $\gamma \subseteq \partial_{i_2-1}^{j_2} \beta_1$. For β_2, the position of i_1 is not changed, therefore $\gamma \subseteq \partial_{i_1}^{j_1} \beta_2$.

□

Lemma 9. *For a homology-suitable collection $\mathcal{F}$ of boxes in $\mathbb{R}^d$ and $\ell = 2, \ldots, d$ we have $\delta_{\ell-1}\delta_\ell = 0$ and $\delta_{\ell-1}^0 \delta_\ell^0 = 0$.*

Proof. Given $\alpha \in \mathcal{F}_\ell$. Then $\delta_{\ell-1}\delta_\ell([\alpha])$ can be written on the form

$$\delta_{\ell-1}\delta_\ell([\alpha]) = \sum_{\gamma \in \mathcal{F}_{\ell-2}} a_\gamma [\gamma] \quad (53)$$

We need to show $a_\gamma = 0$ for all γ. Since $\delta_\ell([\alpha])$ is a sum of elements only on $\partial\alpha$, we must have $\alpha_\gamma = 0$ whenever $\gamma \not\subseteq \partial\alpha$, therefore we can assume $\gamma \subset \partial\alpha$. We can then use Lemma 8, and for the two cases in the lemma, we have either

1. $\alpha_\gamma = (-1)^{i+j}((-1)^{i'+0} + (-1)^{i'+1}) = 0$ or
2. $\alpha_\gamma = (-1)^{i_1+j_1}(-1)^{i_2-1+j_2} + (-1)^{i_2+j_2}(-1)^{i_1+j_1} = 0$

In addition, if $\alpha \in \mathcal{F}^0_\ell$ and $\gamma \in \mathcal{F}^0_{\ell-2}$, then also $\beta_1, \beta_2 \in \mathcal{F}^0_{\ell-1}$ and the same proof works to show $\delta^0_{\ell-1}\delta^0_\ell = 0$. □

Definition 17. *For a homology-suitable collection $\mathcal{F}$ of boxes in $\mathbb{R}^d$, we let $\mathfrak{R}(\mathcal{F})$ and $\mathfrak{R}^0(\mathcal{F})$ be the complexes*

$$\mathfrak{R}(\mathcal{F}) : 0 \xrightarrow{\delta_{d+1}} R_d(\mathcal{F}) \xrightarrow{\delta_d} R_{d-1}(\mathcal{F}) \xrightarrow{\delta_{d-1}} \dots \xrightarrow{\delta_1} R_0(\mathcal{F}) \xrightarrow{\delta_0} 0 \qquad (54)$$

$$\mathfrak{R}^0(\mathcal{F}) : 0 \xrightarrow{\delta^0_{d+1}} R^0_d(\mathcal{F}) \xrightarrow{\delta^0_d} R^0_{d-1}(\mathcal{F}) \xrightarrow{\delta^0_{d-1}} \dots \xrightarrow{\delta^0_1} R^0_0(\mathcal{F}) \xrightarrow{\delta^0_0} 0 \qquad (55)$$

where δ_{d+1}, δ_0, δ^0_{d+1} and δ^0_0 are the zero mappings. These are chain complexes by Lemma 9.

It will be convenient to extend $\mathfrak{R}$ with an extra module, because then all homology classes turn out to be trivial.

Definition 18. *Let $\mathcal{F}$ be a homology-suitable collection of boxes in $\mathbb{R}^d$. We define the module $R_{-1}(\mathcal{F}) = \mathbb{R}$, and the mapping*

$$\begin{aligned} \widetilde{\delta_0} : & R_0(\mathcal{F}) \to R_{-1}(\mathcal{F}) \\ & \sum_{\beta \in \mathcal{F}_0} a_\beta[\beta] \mapsto \sum_\beta a_\beta \end{aligned} \qquad (56)$$

Lemma 10. *For a homology-suitable box collection in $\mathbb{R}^d$, we have $\widetilde{\delta_0}\delta_1 = 0$*

Proof. Any line segment $\beta \in \mathcal{F}_1$ has two endpoints $\gamma_1, \gamma_2 \in \mathcal{F}_0$, where we have $\delta_1([\beta]) = [\gamma_1] - [\gamma_2]$. Then $\widetilde{\delta_0}\delta_1([\beta]) = 1 - 1 = 0$. □

With this, we now define the extended chain complex $\widetilde{\mathfrak{R}}$.

Definition 19. *For a homology-suitable collection $\mathcal{F}$ of boxes in $\mathbb{R}^d$, we let $\widetilde{\mathfrak{R}}(\mathcal{F})$ be the chain complex*

$$\widetilde{\mathfrak{R}}(\mathcal{F}) : 0 \xrightarrow{\delta_{d+1}} R_d(\mathcal{F}) \xrightarrow{\delta_d} \dots \xrightarrow{\delta_1} R_0(\mathcal{F}) \xrightarrow{\widetilde{\delta_0}} R_{-1}(\mathcal{F}) \xrightarrow{\widetilde{\delta_{-1}}} 0 \tag{57}$$

where $\widetilde{\delta_{-1}}$ is the zero mapping. This is a chain complex by Lemma 9 and Lemma 10. The complex is known as the **reduced homology** *of $\mathfrak{R}$.*

We now determine the dimensions of the homology classes for $\mathfrak{R}(\mathcal{F})$, $\mathfrak{R}^0(\mathcal{F})$ and $\widetilde{\mathfrak{R}}(\mathcal{F})$.

Lemma 11. *Let a homology-suitable box collection $\mathcal{F}$ in $\mathbb{R}^d$, $\beta = J_1 \times \ldots \times J_d \in \mathcal{F}_\ell$ for some $\ell \geq 1$, $k \in \{1, \ldots, d\}$ such that $J_k = [c_k, e_k]$ is nontrivial and a number a such that $c_k < a < e_k$ be given. Define*

$$\gamma = J_1 \times \ldots \times J_{k-1} \times \{a\} \times J_{k+1} \times \ldots \times J_d \tag{58}$$
$$\beta_1 = J_1 \times \ldots \times J_{k-1} \times [c_k, a] \times J_{k+1} \times \ldots \times J_d \tag{59}$$
$$\beta_2 = J_1 \times \ldots \times J_{k-1} \times [a, e_k] \times J_{k+1} \times \ldots \times J_d \tag{60}$$

and assume $\gamma \notin \mathcal{F}_{\ell-1}$ and (only when $\ell \geq 2$) $\exists \alpha_1, \ldots, \alpha_n \in \mathcal{F}_{\ell-2}$ such that $\cup_{i=1}^n \alpha_i = \partial\gamma$. Define $\mathcal{F}^+ = (\mathcal{F} \setminus \{\beta\}) \cup \{\gamma, \beta_1, \beta_2\}$. Then $\mathcal{F}^+$ is homology-suitable, and for all $\mathfrak{A} = \mathfrak{R}, \mathfrak{R}^0$ and $\widetilde{\mathfrak{R}}$, we have $H_q(\mathfrak{A}(\mathcal{F})) \cong H_q(\mathfrak{A}(\mathcal{F}^+))$ for all q.

See Figure 4 for some examples where β is replaced by β_1, β_2 and γ.

Proof of Lemma 11. We first show that $\mathcal{F}^+$ is homology-suitable. For the points in Definition 5 we have

1. This is already fullfilled for all boxes in $\mathcal{F}^+ \setminus \{\gamma, \beta_1, \beta_2\}$ ($(\ell+1)$-boxes that have β in the boundary replace this with β_1 and β_2), and also for γ from the lemma. Let $\gamma_1, \ldots, \gamma_n \in \mathcal{F}_{\ell-1}$ be the $(\ell-1)$-boxes that cover $\partial\beta$. Suppose the kth component of some γ_i is $[r, s]$ where $r < a < s$. Then there is a $\boldsymbol{q} \in \mathbb{R}^d$ such that $q_k = a$ and $\boldsymbol{q} \in \gamma_i \setminus \partial\gamma_i$. But also $\boldsymbol{q} \in \partial\gamma$, then there is an $\alpha \in \mathcal{F}_{\ell-2}$ such that $\boldsymbol{q} \in \alpha$, then $\alpha \cap (\gamma_i \setminus \partial\gamma_i) \neq \emptyset$, a contradiction to Definition 5. Therefore, we can reorder the γ_i to $\gamma_1, \ldots, \gamma_m$ where the kth component has an end point $\leq a$, and $\gamma_{m+1}, \ldots, \gamma_n$ where the kth component has a start point $\geq a$. Then $\partial\beta_1 = \gamma_1 \cup \ldots \cup \gamma_m \cup \gamma$ and $\partial\beta_2 = \gamma_{m+1} \cup \ldots \cup \gamma_n \cup \gamma$.

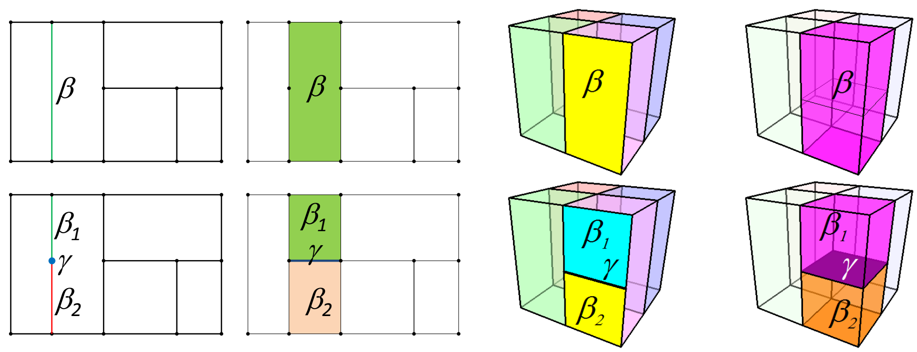


Figure 4: Some examples of a homology-suitable box collection $\mathcal{F}$ in $\mathbb{R}^d$ where an ℓ-box β (top row) is split into two ℓ-boxes β_1 and β_2 together with γ, the $(\ell - 1)$-box between them (bottom row). To the left, for $d = 2$, $\ell = 1$, a line segment in is replaced by to line segments an the point between them. In the second example, $d = \ell = 2$, then a rectangle is replaced by two rectangles and the line γ between them. The end points of γ must already exist in $\mathcal{F}$ (the α_i in Lemma 11). The third example also splits a rectangle, now for $d = 3$. In the fourth example, $d = \ell = 3$, this time a 3-dimensional box is split into two boxes and the rectangle γ between them. In the last example, the boundary lines of γ had to exist in $\mathcal{F}$ prior to the splitting.

2. This is straight forward to check
3. $\mathcal{F}_d$ and $\mathcal{F}_d^+$ are partitions of the same box.

Next we establish the homology isomorphisms. For every $q = 0, \ldots, d$, we define a linear map $\psi_q : R_q(\mathcal{F}) \to R_q(\mathcal{F}^+)$ by $\psi_{\dim\alpha}([\alpha]) = [\alpha]$ for every $\alpha \in \mathcal{F} \setminus \{\beta\}$ while $\psi_\ell([\beta]) = [\beta_1] + [\beta_2]$. ψ_q is injective for every q, and the identity mapping for $q \neq \ell, \ell - 1$, while $R_\ell(\mathcal{F}^+)/\psi_\ell(R_\ell(\mathcal{F})) \cong \mathbb{R}$ and $R_{\ell-1}(\mathcal{F}^+)/\psi_{\ell-1}(R_{\ell-1}(\mathcal{F})) \cong \mathbb{R}$, generated by $[\beta_1]$ and $[\gamma]$ respectively. We now have a commutative diagram with exact sequences on every row

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_{\ell+1}(\mathcal{F}) & \xrightarrow{\psi_{\ell+1}} & R_{\ell+1}(\mathcal{F}^+) & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_\ell(\mathcal{F}) & \xrightarrow{\psi_\ell} & R_\ell(\mathcal{F}^+) & \to & [\beta_1]\mathbb{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\phi & & \\
0 & \to & R_{\ell-1}(\mathcal{F}) & \xrightarrow{\psi_{\ell-1}} & R_{\ell-1}(\mathcal{F}^+) & \to & [\gamma]\mathbb{R} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R_{\ell-2}(\mathcal{F}) & \xrightarrow{\psi_{\ell-2}} & R_{\ell-2}(\mathcal{F}^+) & \to & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{61}$$

where the mapping $\phi : [\beta_1]\mathbb{R} \to [\gamma]\mathbb{R}$ is given as $[\beta_1] \mapsto [\gamma]$ or $[\beta_1] \mapsto -[\gamma]$. The rightmost column is then an exact sequence, but could also be regarded as a chain complex with trivial homology classes. The long exact sequence of homology then gives $H_q(\mathfrak{R}(\mathcal{F})) \cong H_q(\mathfrak{R}(\mathcal{F}^+))$ for all q.

If in addition $\beta \in \mathcal{F}_\ell^0$, then also $\beta_1, \beta_2 \in \mathcal{F}_\ell^0$ and $\gamma \in \mathcal{F}_{\ell-1}^0$, so the same procedure can be used to show $H_q(\mathfrak{R}^0(\mathcal{F})) \cong H_q(\mathfrak{R}^0(\mathcal{F}^+))$ for all q.

The proof also works for $\widetilde{\mathfrak{R}}$ by adding a zero module in the right column for the extra row regarding R_{-1}. □

Lemma 12. *Let $\mathcal{F}$ be a homology-suitable box collection in $\mathbb{R}^d$, and let $\mathcal{H}$ be the trivial homology-suitable box collection on the unit d-cube, given as*

$$\mathcal{H} = \{J_1 \times \ldots \times J_d : J_k = \{0\}, \{1\} \text{ or } [0,1] \text{ for all } k = 1, \ldots, d\} \tag{62}$$

Then for all $\mathfrak{A} = \mathfrak{R}, \mathfrak{R}^0$ and $\widetilde{\mathfrak{R}}$, we have $H_q(\mathfrak{A}(\mathcal{F})) \cong H_q(\mathfrak{A}(\mathcal{H}))$ for all q.

Proof. By linear transformations we can assume that $\mathcal{F}_d$ is a box partition of the unit d-cube $[0,1]^d$. We can extend $\mathcal{F}$ to a full tensor product box collection $\mathcal{F}^T = \mathcal{F}(\mathcal{M})$ for a tensor product mesh $\mathcal{M}$ by making simple extensions as in Lemma 11. The only restriction on the insertion order is that a splitting box γ in Lemma 11 can not be inserted before all of its boundary has been inserted. The same extensions could be done from $\mathcal{H}$ to $\mathcal{F}^T$. By Lemma 11, we have $H_q(\mathfrak{A}(\mathcal{F})) \cong H_q(\mathfrak{A}(\mathcal{F}^T)) \cong H_q(\mathfrak{A}(\mathcal{H}))$ for all q and $\mathfrak{A}$. □

Theorem 1. *Let $\mathcal{F}$ be a homology-suitable box collection in $\mathbb{R}^d$. Then*

$$H_q(\mathfrak{R}(\mathcal{F})) \cong \begin{cases} \mathbb{R} & \text{if } q = 0, \\ 0 & \text{if } q = 1, \dots, d \end{cases} \tag{63}$$

$$H_q(\mathfrak{R}^0(\mathcal{F})) \cong \begin{cases} \mathbb{R} & \text{if } q = d, \\ 0 & \text{if } q = 0, \dots, d-1 \end{cases} \tag{64}$$

$$H_q(\widetilde{\mathfrak{R}}(\mathcal{F})) = 0 \text{ for all } q = -1, \dots, d \tag{65}$$

Proof. By Lemma 12, it sufficies to establish the homology class dimensions for the box partition $\mathcal{H}$.

This is trivial for the $\mathfrak{R}^0$ homologies. The only element in $\mathcal{H}^0$ is the d-box $[0,1]^d$, then $R^0_d(\mathcal{H}) \cong \mathbb{R}$ and $R^0_q(\mathcal{H}) = 0$ for $q < d$, and so $H_d(\mathfrak{R}^0(\mathcal{H})) \cong \mathbb{R}$ and $H_q(\mathfrak{R}^0(\mathcal{H})) = 0$ for $q < d$.

The $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ homologies need more investigation. First notice that for any homology-suitable box collection $\mathcal{F}$, $\widetilde{\delta}_0$ maps $R_0(\mathcal{F})$ surjectively on $R_{-1}(\mathcal{F})$, hence $\dim\ker\widetilde{\delta}_0 = \dim\ker\delta_0 - 1$, while the two complexes are identical before that. Therefore

$$\dim H_q(\widetilde{\mathfrak{R}}(\mathcal{F})) = \begin{cases} \dim H_q(\mathfrak{R}(\mathcal{F})) & \text{if } q \geq 1 \\ \dim H_q(\mathfrak{R}(\mathcal{F})) - 1 & \text{if } q = 0 \end{cases} \tag{66}$$

We put a lexigraphical ordering on $\mathcal{H}$ as follows: For the univariate case, we define $\{0\} < \{1\} < [0,1]$ and in general we define $J_1 \times \dots \times J_d < K_1 \times \dots \times K_d$ if $J_1 = K_1, \dots, J_{k-1} = K_{k-1}$ and $J_k < K_k$ for some k. For $q \geq 1$ define

$$X_q = \{\beta \in \mathcal{H} : \beta = \{1\}^{k-1} \times [0,1] \times J_{k+1} \times \ldots \times J_d \text{ for some } k \geq 1\} \tag{67}$$

i.e. X_q is the set of q-boxes in $\mathcal{H}$ such that there is a $[0,1]$-component that comes before the first (if any) $\{0\}$-component. If $\beta \in \mathcal{H}$ is a box of dimension

≥ 1, it must have at least one $[0,1]$-component, then we define $\Phi(\beta)$ to be the box given by replacing the first $[0,1]$-component of β by $\{0\}$. This induces a bijection between X_{q+1} and $\mathcal{H}_q \setminus X_q$ for $q \geq 1$, and a bijection between X_1 and $\mathcal{H}_0 \setminus \{1\}^d$. Therefore we have

$$\#X_{q+1} = \begin{cases} \#\mathcal{H}_q - \#X_q & \text{if } 1 \leq q \leq d-1 \\ \#\mathcal{H}_q - 1 & \text{if } q = 0 \end{cases} \tag{68}$$

Suppose $q \geq 0$ and given $\beta \in X_{q+1}$. Then $\Phi(\beta)$ is the lexiographically smallest of the faces of β. Because of the injectivity of Φ on X_{q+1}, this must mean that the lexiographically smallest γ in $\mathcal{H}_q$ with a non-zero coefficient in the expression for $\delta_{q+1}([\beta])$ is different for every $\beta \in X_{q+1}$, and therefore the induced mapping

$$\delta_{q+1} : \bigoplus_{\beta \in X_{q+1}} [\beta]\mathbb{R} \to R_q(\mathcal{H}) \tag{69}$$

must be injective, hence

$$\dim \operatorname{im} \delta_{q+1} \geq \#X_{q+1} \tag{70}$$

for all $q = 0, \ldots, d-1$. For $q \geq 1$, (68) and (70) give

$$\begin{aligned} \dim \operatorname{im} \delta_{q+1} \geq \#X_{q+1} = \#\mathcal{H}_q - \#X_q \geq \dim R_q(\mathcal{H}) - \dim \operatorname{im} \delta_q \\ = \dim \ker \delta_q = \dim H_q(\mathfrak{R}(\mathcal{H})) + \dim \operatorname{im} \delta_{q+1} \end{aligned} \tag{71}$$

Comparing the first and last part in this inequality gives $H_q(\mathfrak{R}(\mathcal{H})) = 0$.

For $q = 0$, we have

$$\begin{aligned} \dim \operatorname{im} \delta_1 \geq \#X_1 = \#\mathcal{H}_0 - 1 = \dim \ker \delta_0 - 1 \\ = \dim H_0(\mathfrak{R}(\mathcal{H})) + \dim \operatorname{im} \delta_1 - 1 \end{aligned} \tag{72}$$

hence $\dim H_0(\mathfrak{R}(\mathcal{H})) \leq 1$. With (66) we get $\dim H_0(\mathfrak{R}(\mathcal{H})) = 1$ while $\dim H_q(\mathfrak{R}(\mathcal{H}))$ and $\dim H_{q'}(\widetilde{\mathfrak{R}}(\mathcal{H}))$ are all zero for $q \geq 1$ and $q' \geq 0$. Finally $H_{-1}(\widetilde{\mathfrak{R}}(\mathcal{H})) = 0$ because $\widetilde{\delta_0}$ is surjective. □

5 Homologies on spline meshes

Definition 20. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$.*

- *For a (k, a)-meshrectangle $\beta \in \mathcal{M}$, we define*

$$P_\beta(\mathcal{N}) = (x_k - a)^{p_k - \mu(\beta) + 1} \tag{73}$$

  which is a polynomial in $\Pi_{\boldsymbol{p}}$.

- *For a box $\beta \in \mathcal{F}(\mathcal{M})$, we define the vetor space*

$$\Delta_\beta(\mathcal{N}) = < P_\gamma(\mathcal{N}) : \gamma \in \mathcal{D}_\beta(\mathcal{M}) > \cap \Pi_{\boldsymbol{p}} \tag{74}$$

  where $< f_1, \ldots, f_n >$ is the ideal in Π^d generated by the polynomials f_i. $\Delta_\beta(\mathcal{N})$ is a vector subspace of $\Pi_{\boldsymbol{p}}$.

Notice that if $\dim \beta = d$, then $\Delta_\beta(\mathcal{N}) = 0$. If β is a (k, a)-meshrectangle, then $\Delta_\beta(\mathcal{N})$ is the set of all polynomials $P_\beta(\mathcal{N}) \cdot F$ for polynomials F of degree up to $\mu(\beta) - 1$ in x_k and up to p_i in x_i, $i \neq k$. In general for $\beta \in \mathcal{F}(\mathcal{M})$, we have

$$\Delta_\beta(\mathcal{N}) = \sum_{\gamma \in \mathcal{D}_\beta(\mathcal{M})} \Delta_\gamma(\mathcal{N}) \tag{75}$$

We also have, for any point $\boldsymbol{q} \in \beta$,

$$\Delta_\beta(\mathcal{N}) = < (x_k - q_k)^{p_k - \mu_k(\beta) + 1} : k = 1, \ldots, d > \tag{76}$$

Also notice that for boxes $\beta \subseteq \gamma$ in $\mathcal{F}(\mathcal{M})$ we have $\Delta_\gamma(\mathcal{N}) \subseteq \Delta_\beta(\mathcal{N})$.

Definition 21. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh in $\mathbb{R}^d$. For $\ell = 0, \ldots, d$, we define the modules*

$$I_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}_\ell(\mathcal{M})} [\beta] \Delta_\beta(\mathcal{N}) \tag{77}$$

$$R_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}_\ell(\mathcal{M})} [\beta] \Pi_{\boldsymbol{p}} \tag{78}$$

$$S_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}_\ell(\mathcal{M})} [\beta] \Pi_{\boldsymbol{p}} / \Delta_\beta(\mathcal{N}) \tag{79}$$

and we define the linear maps $\delta_\ell : I_\ell(\mathcal{N}) \to I_{\ell-1}(\mathcal{N})$, $\delta_\ell : R_\ell(\mathcal{N}) \to R_{\ell-1}(\mathcal{N})$ and $\delta_\ell : S_\ell(\mathcal{N}) \to S_{\ell-1}(\mathcal{N})$ by

$$\delta_\ell(\sum_\beta f_\beta[\beta]) = \sum_\beta f_\beta \delta_\ell([\beta]) \tag{80}$$

where we reuse the definition of δ_ℓ *from* (46). *The exact sequences*

$$0 \to \Delta_\beta(\mathcal{N}) \to \Pi_{\boldsymbol{p}} \to \Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N}) \to 0 \tag{81}$$

for each $\beta \in \mathcal{F}(\mathcal{N})$ *give rise to exact sequences*

$$0 \to I_\ell(\mathcal{N}) \to R_\ell(\mathcal{N}) \to S_\ell(\mathcal{N}) \to 0 \tag{82}$$

for each $\ell = 0, \ldots, d$. *These take part in the following commutative diagram, where the rows define the chain complexes* $\mathfrak{I}(\mathcal{N})$, $\mathfrak{R}(\mathcal{N})$ *and* $\mathfrak{S}(\mathcal{N})$.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{I}(\mathcal{N}) : 0 & \xrightarrow{\delta_{d+1}} & I_d(\mathcal{N}) & \xrightarrow{\delta_d} & I_{d-1}(\mathcal{N}) & \xrightarrow{\delta_{d-1}} & \ldots & \xrightarrow{\delta_1} & I_0(\mathcal{N}) & \xrightarrow{\delta_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{R}(\mathcal{N}) : 0 & \xrightarrow{\delta_{d+1}} & R_d(\mathcal{N}) & \xrightarrow{\delta_d} & R_{d-1}(\mathcal{N}) & \xrightarrow{\delta_{d-1}} & \ldots & \xrightarrow{\delta_1} & R_0(\mathcal{N}) & \xrightarrow{\delta_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{S}(\mathcal{N}) : 0 & \xrightarrow{\delta_{d+1}} & S_d(\mathcal{N}) & \xrightarrow{\delta_d} & S_{d-1}(\mathcal{N}) & \xrightarrow{\delta_{d-1}} & \ldots & \xrightarrow{\delta_1} & S_0(\mathcal{N}) & \xrightarrow{\delta_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array} \tag{83}$$

By the diagram in (83), we now get the long exact sequence of homology classes

$$\begin{aligned}
0 \to & H_d(\mathfrak{I}(\mathcal{N})) \to H_d(\mathfrak{R}(\mathcal{N})) \to H_d(\mathfrak{S}(\mathcal{N})) \to H_{d-1}(\mathfrak{I}(\mathcal{N})) \to \ldots \\
& \ldots \to H_q(\mathfrak{I}(\mathcal{N})) \to H_q(\mathfrak{R}(\mathcal{N})) \to H_q(\mathfrak{S}(\mathcal{N})) \to \ldots \\
& \ldots \to H_1(\mathfrak{S}(\mathcal{N})) \to H_0(\mathfrak{I}(\mathcal{N})) \to H_0(\mathfrak{R}(\mathcal{N})) \to H_0(\mathfrak{S}(\mathcal{N})) \to 0
\end{aligned} \tag{84}$$

We will need the reduced homologies from the chain complexes $\widetilde{\mathfrak{I}}$ and $\widetilde{\mathfrak{R}}$.

Definition 22. *Let* $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ *be a spline mesh in* $\mathbb{R}^d$. *Define the modules* $I_{-1}(\mathcal{N}) = \Pi_{\boldsymbol{p}}$ *and* $R_{-1}(\mathcal{N}) = \Pi_{\boldsymbol{p}}$, *and boundary maps* $\widetilde{\delta_{0,I}} : I_0(\mathcal{N}) \to I_{-1}(\mathcal{N})$ *and* $\widetilde{\delta_{0,R}} : R_0(\mathcal{N}) \to R_{-1}(\mathcal{N})$ *by*

$$\widetilde{\delta_0}(\sum_\beta f_\beta[\beta]) = \sum_\beta f_\beta \tag{85}$$

reusing the $\widetilde{\delta_0}$ *in* (56). *By Lemma 10, we get chain complexes*

$$\widetilde{\mathfrak{I}}(\mathcal{N}) : 0 \xrightarrow{\delta_{d+1}} I_d(\mathcal{N}) \xrightarrow{\delta_d} \ldots \xrightarrow{\delta_1} I_0(\mathcal{N}) \xrightarrow{\widetilde{\delta_0}} I_{-1}(\mathcal{N}) \xrightarrow{\delta_{-1}} 0 \tag{86}$$

$$\widetilde{\mathfrak{R}}(\mathcal{N}) : 0 \xrightarrow{\delta_{d+1}} R_d(\mathcal{N}) \xrightarrow{\delta_d} \ldots \xrightarrow{\delta_1} R_0(\mathcal{N}) \xrightarrow{\widetilde{\delta_0}} R_{-1}(\mathcal{N}) \xrightarrow{\delta_{-1}} 0 \tag{87}$$

For the homologies of $\mathfrak{I}$ and $\mathfrak{R}$ we have $H_q(\widetilde{\mathfrak{I}}(\mathcal{N})) = H_q(\mathfrak{I}(\mathcal{N}))$ and $H_q(\widetilde{\mathfrak{R}}(\mathcal{N})) = H_q(\mathfrak{R}(\mathcal{N}))$ for $q \geq 1$.

If we put $S_{-1}(\mathcal{N}) = 0$ and use the identity mapping between $I_{-1}(\mathcal{N})$ and $R_{-1}(\mathcal{N})$, we can extend the diagram in (83) to get a long exact sequence

$$\begin{aligned} 0 \rightarrow & H_d(\mathfrak{I}(\mathcal{N})) \rightarrow H_d(\mathfrak{R}(\mathcal{N})) \rightarrow H_d(\mathfrak{S}(\mathcal{N})) \rightarrow H_{d-1}(\mathfrak{I}(\mathcal{N})) \rightarrow \dots \\ & \dots \rightarrow H_1(\mathfrak{S}(\mathcal{N})) \rightarrow H_0(\widetilde{\mathfrak{I}}(\mathcal{N})) \rightarrow H_0(\widetilde{\mathfrak{R}}(\mathcal{N})) \\ & \rightarrow H_0(\mathfrak{S}(\mathcal{N})) \rightarrow H_{-1}(\widetilde{\mathfrak{I}}(\mathcal{N})) \rightarrow H_{-1}(\widetilde{\mathfrak{R}}(\mathcal{N})) \rightarrow 0 \end{aligned} \tag{88}$$

The third set of chain complexes we will look at are the $\mathfrak{I}^0$, $\mathfrak{R}^0$ and $\mathfrak{S}^0$ chain complexes.

Definition 23. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh in $\mathbb{R}^d$. For $\ell = 0, \dots, d$, we define the modules*

$$I^0_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}^0_\ell(\mathcal{M})} [\beta]\Delta_\beta(\mathcal{N}) \tag{89}$$

$$R^0_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}^0_\ell(\mathcal{M})} [\beta]\Pi_{\boldsymbol{p}} \tag{90}$$

$$S^0_\ell(\mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}^0_\ell(\mathcal{M})} [\beta]\Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N}) \tag{91}$$

and we define the linear maps $\delta^0_\ell : I^0_\ell(\mathcal{N}) \rightarrow I^0_{\ell-1}(\mathcal{N})$, $\delta^0_\ell : R^0_\ell(\mathcal{N}) \rightarrow R^0_{\ell-1}(\mathcal{N})$ and $\delta^0_\ell : S^0_\ell(\mathcal{N}) \rightarrow S^0_{\ell-1}(\mathcal{N})$ just as for δ_ℓ by

$$\delta^0_\ell(\sum_\beta f_\beta[\beta]) = \sum_\beta f_\beta \delta^0_\ell([\beta]) \tag{92}$$

Again, we get a diagram with exact sequences on the columns, and where the *rows give three new chain complexes:*

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{I}^0(\mathcal{N}) : 0 & \xrightarrow{\delta^0_{d+1}} & I^0_d(\mathcal{N}) & \xrightarrow{\delta^0_d} & I^0_{d-1}(\mathcal{N}) & \xrightarrow{\delta^0_{d-1}} & \dots & \xrightarrow{\delta^0_1} & I^0_0(\mathcal{N}) & \xrightarrow{\delta^0_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{R}^0(\mathcal{N}) : 0 & \xrightarrow{\delta^0_{d+1}} & R^0_d(\mathcal{N}) & \xrightarrow{\delta^0_d} & R^0_{d-1}(\mathcal{N}) & \xrightarrow{\delta^0_{d-1}} & \dots & \xrightarrow{\delta^0_1} & R^0_0(\mathcal{N}) & \xrightarrow{\delta^0_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{S}^0(\mathcal{N}) : 0 & \xrightarrow{\delta^0_{d+1}} & S^0_d(\mathcal{N}) & \xrightarrow{\delta^0_d} & S^0_{d-1}(\mathcal{N}) & \xrightarrow{\delta^0_{d-1}} & \dots & \xrightarrow{\delta^0_1} & S^0_0(\mathcal{N}) & \xrightarrow{\delta^0_0} & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}
\tag{93}
$$

Of course, again, we get a long exact sequence

$$
\begin{aligned}
0 \to & H_d(\mathfrak{I}^0(\mathcal{N})) \to H_d(\mathfrak{R}^0(\mathcal{N})) \to H_d(\mathfrak{S}^0(\mathcal{N})) \to H_{d-1}(\mathfrak{I}^0(\mathcal{N})) \to \dots \\
& \dots \to H_q(\mathfrak{I}^0(\mathcal{N})) \to H_q(\mathfrak{R}^0(\mathcal{N})) \to H_q(\mathfrak{S}^0(\mathcal{N})) \to \dots \\
& \dots \to H_0(\mathfrak{I}^0(\mathcal{N})) \to H_0(\mathfrak{R}^0(\mathcal{N})) \to H_0(\mathfrak{S}^0(\mathcal{N})) \to 0
\end{aligned}
\tag{94}
$$

We now establish some properties of the introduced homolgies

Lemma 13. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh in $\mathbb{R}^d$. Then*

1. $H_q(\mathfrak{R}(\mathcal{N})) = 0$ *for* $1 \le q \le d$ *and* $H_0(\mathfrak{R}(\mathcal{N})) \cong \Pi_{\boldsymbol{p}}$.
2. $H_q(\mathfrak{R}^0(\mathcal{N})) = 0$ *for* $0 \le q \le d-1$ *and* $H_d(\mathfrak{R}^0(\mathcal{N})) \cong \Pi_{\boldsymbol{p}}$.
3. $H_q(\widetilde{\mathfrak{R}}(\mathcal{N})) = 0$ *for* $-1 \le q \le d$.
4. $H_d(\mathfrak{I}(\mathcal{N}) = H_d(\mathfrak{I}^0(\mathcal{N}) = H_d(\widetilde{\mathfrak{I}}(\mathcal{N}) = 0$.
5. $H_q(\mathfrak{S}(\mathcal{N})) \cong H_{q-1}(\mathfrak{I}(\mathcal{N}))$ *for* $q = 2, \dots, d$.
6. $H_q(\mathfrak{S}(\mathcal{N})) \cong H_{q-1}(\widetilde{\mathfrak{I}}(\mathcal{N}))$ *for all* $q = 0, \dots, d$.
7. $H_q(\mathfrak{S}^0(\mathcal{N})) \cong H_{q-1}(\mathfrak{I}^0(\mathcal{N}))$ *for* $q = 0, \dots, d-1$.
8. *If $\mathcal{N}$ is an open spline mesh,* $H_q(\mathfrak{S}(\mathcal{N})) \cong H_q(\mathfrak{S}^0(\mathcal{N}))$ *for all* $q = 0, \dots, d$, *and in particular* $H_0(\mathfrak{S}^0(\mathcal{N}) = H_0(\mathfrak{S}(\mathcal{N})) = 0$.

Proof.

1.-3. The $\mathfrak{R}(\mathcal{N})$, $\mathfrak{R}^0(\mathcal{N})$ and $\widetilde{\mathfrak{R}}(\mathcal{N})$ chain complexes are built in the same way as for $\mathfrak{R}(\mathcal{F})$ etc for homology-suitable box collections, except that we use the vector space $\Pi_{\boldsymbol{p}}$ instead of $\mathbb{R}$. Therefore, the result is given by Theorem 1.

4. Because $\Delta_\beta(\mathcal{N}) = 0$ when β is d-dimensional, the modules $I_d(\mathcal{N})$ and $I_d^0(\mathcal{N})$ are zero.

5-7. From the long exact sequence of homologies, we have en exact sequence

$$H_q(\mathfrak{R}(\mathcal{N})) \to H_q(\mathfrak{S}(\mathcal{N})) \to H_{q-1}(\mathfrak{I}(\mathcal{N})) \to H_{q-1}(\mathfrak{R}(\mathcal{N})) \tag{95}$$

so whenever $H_q(\mathfrak{R}(\mathcal{N})) = H_{q-1}(\mathfrak{R}(\mathcal{N})) = 0$ we get $H_q(\mathfrak{S}(\mathcal{N})) \cong H_{q-1}(\mathfrak{I}(\mathcal{N}))$. This also aplies for the $\mathfrak{R}^0(\mathcal{N})$ and $\widetilde{\mathfrak{R}}(\mathcal{N})$ chain complexes.

8. If γ is a (k, a)-meshrectangle at the boundary $\partial\Omega(\mathcal{M})$, we have $\mu(\gamma) = p_k + 1$ because $\mathcal{N}$ is an open spline mesh. Then $\Delta_\gamma(\mathcal{N})$ is generated by the polynomial $(x_k - a_k)^{p_k - \mu(\gamma)+1} = 1$, i.e. $\Delta_\gamma(\mathcal{N}) = \Pi_{\boldsymbol{p}}$. For a general box $\beta \in \mathcal{F}(\mathcal{M}) \setminus \mathcal{F}^0(\mathcal{M})$ on $\partial\Omega(\mathcal{M})$, we must have $\beta \subseteq \gamma$ for some meshrectangle $\gamma \subseteq \partial\Omega(\mathcal{M})$ then $\Delta_\beta(\mathcal{N}) = \Pi_{\boldsymbol{p}}$ because $\Pi_{\boldsymbol{p}} = \Delta_\gamma(\mathcal{N}) \subseteq \Delta_\beta(\mathcal{N})$. Therefore $\Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N}) = 0$ and so $S_q(\mathcal{N}) \cong S_q^0(\mathcal{N})$ for all q. This isomorphism commutes with the boundary maps δ_q, hence $H_q(\mathfrak{S}(\mathcal{N})) \cong H_q(\mathfrak{S}^0(\mathcal{N}))$. One of the long exact sequences of homologies ends with $H_0(\mathfrak{R}^0(\mathcal{N})) \to H_0(\mathfrak{S}^0(\mathcal{N})) \to 0$ where $H_0(\mathfrak{R}^0(\mathcal{N})) = 0$, therefore $H_0(\mathfrak{S}^0(\mathcal{N})) = 0$.

□

6 The dimension formula

Lemma 14. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh in $\mathbb{R}^d$. The dimension of the spline space $\mathbb{S}(\mathcal{N})$ is then given as*

$$\dim \mathbb{S}(\mathcal{N}) = \dim H_d(\mathfrak{S}(\mathcal{N})) \tag{96}$$

If the mesh is open, the dimension of the spline space is also given as

$$\dim \mathbb{S}(\mathcal{N}) = \dim H_d(\mathfrak{S}^0(\mathcal{N})) \tag{97}$$

Proof. Given a function f on $\mathbb{R}^d$ which is zero outside $\Omega(\mathcal{M})$ and given by a polynomial $f_\beta \in \Pi_{\boldsymbol{p}}$ on each d-box $\beta \in \mathcal{F}_d(\mathcal{M})$. It is a natural element of $R_d(\mathcal{N})$, this is the same as $S_d(\mathcal{N})$ because $I_d(\mathcal{N}) = 0$. For each inner (k, a)-meshrectangle $\gamma \in \mathcal{F}_{d-1}(\mathcal{M})$, there are two boxes β_1 and β_2 containing γ. They are given by extending γ in the kth variable in positive or negative direction. Then $\gamma \subseteq \partial_i^0 \beta_1$ and $\gamma \subseteq \partial_i^1 \beta_2$ for some i. The only contribution from $\delta_q(f)$ to the component $[\gamma]$ comes from $[\beta_1]$ and $[\beta_2]$ and must therefore be $\pm(f_{\beta_1} - f_{\beta_2})$. This is zero in $S_{d-1}(\mathcal{N})$ if and only if $f_{\beta_1} - f_{\beta_2}$ is divisible by $(x_k - a)^{p_k - \mu(\gamma)+1}$ which is if and only if f has $C^{p_k-\mu(\gamma)}$-continuity along γ. For a boundary meshrectangle γ, the same argument shows that f has $C^{p_k-\mu(\gamma)}$-continuity along γ if and only if the component of $[\gamma]$ in $\delta_q(f)$ is zero. Thus, f is a spline function in $\mathbb{S}(\mathcal{N})$ if and only if $\delta_q(f) = 0$. As $\operatorname{im} \delta_{d+1} = 0$, we have $H_q(\mathfrak{S}(\mathcal{N})) \cong \ker \delta_q \cong \mathbb{S}(\mathcal{N})$. By Lemma 13, this is also isomorphic to $H_q(\mathfrak{S}^0(\mathcal{N}))$ if $\mathcal{N}$ is open. $\square$

This gives a simple expression for the spline space dimension, but it involves a homology class which is non-zero. There is a more important formula that gives the dimension based on combinatorial countings and homolgy terms, where the homolgies in many cases become zero as we will see later.

Theorem 2. *Let $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ be a spline mesh in $\mathbb{R}^d$. Then the dimension of the spline space $\mathbb{S}(\mathcal{N})$ is given by*

$$\begin{aligned}\dim \mathbb{S}(\mathcal{N}) = &\sum_{\ell=0}^{d} (-1)^{d-\ell} \left(\sum_{\beta \in \mathcal{F}_\ell(\mathcal{M})} \prod_{k=1}^{d} \big(p_k - \mu_k(\beta) + 1\big) \right) \\ &- \sum_{q=0}^{d-1} (-1)^{d-q} \dim H_q(\mathfrak{S}(\mathcal{N}))\end{aligned} \tag{98}$$

If $\mathcal{N}$ is open, the dimension is also given as

$$\begin{aligned}\dim \mathbb{S}(\mathcal{N}) = &\sum_{\ell=0}^{d} (-1)^{d-\ell} \left(\sum_{\beta \in \mathcal{F}_\ell^0(\mathcal{M})} \prod_{k=1}^{d} \big(p_k - \mu_k(\beta) + 1\big) \right) \\ &- \sum_{q=1}^{d-1} (-1)^{d-q} \dim H_q(\mathfrak{S}^0(\mathcal{N}))\end{aligned} \tag{99}$$

Proof. By Lemma 6,

$$\sum_{q=0}^{d}(-1)^{d-q}\dim H_q(\mathfrak{S}(\mathcal{N})) = \sum_{\ell=0}^{d}(-1)^{d-\ell}\dim S_\ell(\mathcal{N})$$
$$= \sum_{\ell=0}^{d}(-1)^{d-\ell}\left(\sum_{\beta\in\mathcal{F}_\ell(\mathcal{M})}\dim(\Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N}))\right) \tag{100}$$

If, for simplicity, we assume the trivial components of β are $\{0\}$, a basis for the module $\Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N})$ is given by the monomials $x_1^{i_1}\dots x_d^{i_d}$ where $0 \le i_k \le p_k - \mu_k(\beta)$. By Lemma 14 we then have

$$\dim \mathbb{S}(\mathcal{N}) = \sum_{\ell=0}^{d}(-1)^{d-\ell}\left(\sum_{\beta\in\mathcal{F}_\ell(\mathcal{M})}\prod_{k=1}^{d}\big(p_k-\mu_k(\beta)+1\big)\right) + \sum_{q=0}^{d-1}(-1)^{d-q+1}\dim H_q(\mathfrak{S}(\mathcal{N})) \tag{101}$$

If $\mathcal{N}$is open, we use the same method on the $\mathfrak{S}^0$ chain complex, and we can omit the term $H_0(\mathfrak{S}^0(\mathcal{N}))$ as this is zero by Lemma 13. □

For a boxmesh $\mathcal{M}$ in $\mathbb{R}^d$ and a subset $B \subseteq \{1,\dots,d\}$ we define $f_B^0(\mathcal{M}) = \#\mathcal{F}_B$. Then we get a special case of the spline dimension formula when all inner meshrectangles in the same direction have the same multiplicity:

Corollary 1. *Let $\mathcal{N} = (\mathcal{M},\mu,\boldsymbol{p})$ be an open spline mesh in $\mathbb{R}^d$ and $m_1,\dots,m_d$ numbers such that $\mu(\beta) = m_k$ for every inner k-meshrectangle β. Then the dimension of the spline space $\mathbb{S}(\mathcal{N})$ is given by*

$$\dim \mathbb{S}(\mathcal{N}) = \sum_{B\subseteq\{1,\dots,d\}}(-1)^{d-|B|}f_B^0(\mathcal{M})\prod_{k\in B}(p_k+1)\prod_{k\notin B}(p_k-m_k+1) - \sum_{q=1}^{d-1}(-1)^{d-q}\dim H_q(\mathfrak{S}^0(\mathcal{N})) \tag{102}$$

The spline formulas all consist of two parts. The first part is a combinatorial counting, and is easy to do by a computer holding the topological represenation. It is independent of the parameterization, and does not involve any deeper linear algebra.

The second part is harder to calculate. It is a sum of dimensions homolgoy terms, and might even depend on the parameterization. The rest of this paper will basically focus on how to determine, or at least put an upper bound on these homology terms.

7 The homology term for tensor product splines

We start the investigation of the homologies from a spline mesh by looking at the univariate case.

Lemma 15. *Let $\mathcal{N}(\boldsymbol{a}, \boldsymbol{\mu}, p)$ be a univariate spline mesh, where $\boldsymbol{a}$, $\boldsymbol{\mu}$ and p are given as in Definition 13. If we let*

$$\alpha = \sum_{i=1}^{n} \mu_i - (p+1) \tag{103}$$

we have

$$\dim H_0(\mathfrak{S}(\mathcal{N})) = \max(0, -\alpha) \tag{104}$$

$$\dim H_1(\mathfrak{S}(\mathcal{N})) = \dim \mathbb{S}(\mathcal{N}) = \max(0, \alpha) \tag{105}$$

Proof. The set $\mathcal{F}_1(\mathcal{M})$ is the set of all intervals $[a_i, a_{i+1}]$ for $i = 1, \ldots, n-1$, and for any $\beta \in \mathcal{F}_1(\mathcal{M})$, we have $\Delta_\beta(\mathcal{N}) = 0$, then $\dim(\Pi_p/\Delta_\beta(\mathcal{N})) = \dim \Pi_p = p+1$. Therefore $\dim S_1(\mathcal{N}) = (n-1)(p+1)$. The set $\mathcal{F}_0(\mathcal{M})$ is the set of all points given by $x = a_i$ for $i = 1, \ldots, n$. For $\beta = \{a_i\}$, $\Delta_\beta(\mathcal{N})$ is the set of all polynomials in Π_p divisible by $(x - a_i)^{p+1-\mu_i}$, then $\dim(\Pi_p/\Delta_\beta(\mathcal{N})) = p + 1 - \mu_i$, generated by the basis $\{1, x - a_i, \ldots, (x - a_i)^{p-\mu_i}\}$. Hence $\dim S_0(\mathcal{N}) = \sum_{i=1}^{n}(p+1-\mu_i)$. Therefore

$$\begin{aligned} \dim H_1(\mathfrak{S}(\mathcal{N})) - \dim H_0(\mathfrak{S}(\mathcal{N})) &= \dim S_1(\mathcal{N}) - \dim S_0(\mathcal{N}) \\ &= (n-1)(p+1) - \sum_{i=1}^{n}(p+1-\mu_i) = \alpha \end{aligned} \tag{106}$$

It is well known that the spline space of p-degree functions defined by a knot vector of length m has dimension $m-(p+1)$ if $m \geq p+1$, and is trivial if $m \leq p+1$. Therefore $\dim \mathbb{S}(\mathcal{N}) = \max(0, \alpha)$, this is the same as $\dim H_1(\mathfrak{S}(\mathcal{N}))$. Then $\dim H_0(\mathfrak{S}(\mathcal{N})) = \dim H_1(\mathfrak{S}(\mathcal{N})) - \alpha = \max(0, -\alpha)$ □

We move to general tensor product spline meshes. Let $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_d$ be a tensor product spline mesh where each $\mathcal{N}_k$ is given as $\mathcal{N}(\boldsymbol{a}_k, \boldsymbol{\mu}_k, pk)$ as in Definition 13. The underlying mesh $\mathcal{M}$ of $\mathcal{N}$ is given by $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_d$ where ech $\mathcal{M}_k$ is the one-dimensional mesh $\mathcal{M}(\boldsymbol{a}_k)$.

Given an (ℓ, d) box $\beta = J_1 \times \ldots \times J_d \in \mathcal{F}_\ell(\mathcal{M})$. For each k, either $J_k = \{a_{k,j}\} \in \mathcal{F}_0(\mathcal{M}_k)$ is a point, then let $\nu_k = \mu_{k,j}$ and $b_k = a_{k,j}$, or $J_k \in \mathcal{F}_1(\mathcal{M}_k)$ is a non-trivial segment, then let $\nu_k = 0$ and $b_k = 0$. For every k, the group $\Delta_{J_k}(\mathcal{N}_k)$ is given as the set of polynomials in $\Pi_{(p_k)}$ divisible by $(x_k - b_k)^{p_k+1-\nu_k}$. The group $\Delta_\beta(\mathcal{N})$ is the set of all polynomials in $\Pi_{\boldsymbol{p}}$ from the ideal in Π^d generated by $(x_k - b_k)^{p_k+1-\nu_k}$ for all k. This gives a natural isomorphism

$$\begin{aligned} \Pi_{\boldsymbol{p}}/\Delta_{J_1\times\ldots\times J_k}(\mathcal{N}) &\cong \bigotimes_{k=1}^{d} \Pi_{(p_k)}/\Delta_{J_k}(\mathcal{N}_k) \\ \prod_{k=1}^{d}(x_k - b_k)^{n_k} &\mapsto \bigotimes_{k=1}^{d}(x_k - b_k)^{n_k} \end{aligned} \tag{107}$$

For a set $B \subset \{1, \ldots, d\}$, let $\chi_B : \{1, \ldots, d\} \to \{0, 1\}$ be the caracteristic function of B sending k to 1 if and only if $k \in B$. If $\#B = q$, we define the set

$$\mathcal{F}_B(\mathcal{M}) := \{J_1 \times \ldots \times J_d \in \mathcal{F}_q(\mathcal{M}) \ : J_k \in \mathcal{F}_{\chi_B(k)}(\mathcal{M}_k) \forall k\} \tag{108}$$

The isomorphism (107) and the distributive property of tensor products over direct sums yields

$$\begin{aligned} \bigoplus_{\beta \in \mathcal{F}_B(\mathcal{N})} [\beta] \Pi_{\boldsymbol{p}}/\Delta_\beta(\mathcal{N}) &\cong \bigoplus_{\substack{\beta = J_1\times\ldots\times J_d \\ J_k \in \mathcal{F}_{\chi_B}(k)}} [\beta] \left(\bigotimes_{k=1}^{d} \Pi_{(p_k)}/\Delta_{J_k}(\mathcal{N}_k) \right) \\ &\cong \bigotimes_{k=1}^{d} S_{\chi_B(k)}(\mathcal{N}_k) \end{aligned} \tag{109}$$

If we combine this for all sets B of the same size, we have

$$S_q(\mathcal{N}) \cong \bigoplus_{\substack{B \subset \{1,\ldots,d\} \\ \#B = q}} \left(\bigotimes_{k=1}^{d} S_{\chi_B(k)}(\mathcal{N}_k) \right) \tag{110}$$

We now describe the boundary map $\delta_q : S_q(\mathcal{N}) \to S_{q-1}(\mathcal{N})$ as a map between the right hand sides in (110). Given $B \subset \{1, \ldots, d\}$ such that $\#B = q$. Given $J_k = [J_k^0, J_k^1] \in \mathcal{F}_{\chi_B(k)}$ and $f_k \in \Pi_{(p_k)}/\Delta_{J_k}(\mathcal{N}_k)$ for all $k = 1, \ldots, d$. Let $i_B(1) < i_B(2) < \ldots < i_B(q)$ be the elements in B. Then

$$\begin{aligned}
&\delta_q([J_1]f_1 \otimes \ldots \otimes [J_d]f_d) \\
&= \sum_{j=1}^{q} \sum_{i=0}^{1} (-1)^{i+j} [J_1]f_1 \otimes \ldots \otimes [J^i_{i_B(j)}] f_{i_B(j)} \otimes \ldots \otimes [J_d]f_d \\
&= \sum_{j=1}^{q} (-1)^{j+1} [J_1]f_1 \otimes \ldots \otimes \delta_1^{i_B(j)}([J_{i_B(j)}]f_{i_B(j)}) \otimes \ldots \otimes [J_d]f_d
\end{aligned} \tag{111}$$

where δ_1^k is the boundary map from $S_1(\mathcal{N}_k)$ to $S_0(\mathcal{N}_k)$. This combines to

$$\delta_q(g_1 \otimes \ldots \otimes g_d) = \sum_{j=1}^{q} (-1)^{j+1} g_1 \otimes \ldots \otimes \delta_1^{i_B(j)}(g_{i_B(j)}) \otimes \ldots \otimes g_d \tag{112}$$

whenever $g_k \in S_{\chi_B(k)}(\mathcal{N}_k)$ for all k.

We can now link the chain complexes of the tensor product spline spaces to tensor product chain complexes. Let $\mathcal{N}_{[a,b]}$ be the $(b-a+1)$-dimensional tensor product mesh we get in the natural way by combining the univariate meshes $\mathcal{N}^k$ for $a \le k \le b$, thus $\mathcal{N}$ that we have been working with so far is the same as $\mathcal{N}_{[1,d]}$.

Lemma 16. *In the situation above, when $d \ge 2$, we have a natural isomorphism*

$$\phi_q : (\mathfrak{S}(\mathcal{N}_{[1,d]}))_q \to (\mathfrak{S}(\mathcal{N}_1) \otimes \mathfrak{S}(\mathcal{N}_{[2,d]}))_q \tag{113}$$

for all q. Also, the boundary maps

$$\delta_q : (\mathfrak{S}(\mathcal{N}_{[1,d]}))_q \to (\mathfrak{S}(\mathcal{N}_{[1,d]}))_{q-1} \tag{114}$$

$$\delta'_q : (\mathfrak{S}(\mathcal{N}_1) \otimes \mathfrak{S}(\mathcal{N}_{[2,d]}))_q \to (\mathfrak{S}(\mathcal{N}_1) \otimes \mathfrak{S}(\mathcal{N}_{[2,d]}))_{q-1} \tag{115}$$

commute with this isomorphism, i.e. $\delta'_q \phi_q = \phi_{q-1} \delta_q$.

Proof. We have

$$
\begin{aligned}
(\mathfrak{S}(\mathcal{N}_{[1,d]}))_q = S_q(\mathcal{N}_{[1,d]}) &\cong \bigoplus_{\substack{B\subset\{1,\ldots,d\}\\ \#B=q}} \left(\bigotimes_{k=1}^{d} S_{\chi_B(k)}(\mathcal{N}_k) \right) \\
&\cong \left(S_0(\mathcal{N}_1) \otimes \bigoplus_{\substack{B\subset\{2,\ldots,d\}\\ \#B=q}} \left(\bigotimes_{k=2}^{d} S_{\chi_B(k)}(\mathcal{N}_k) \right) \right) \otimes \\
&\quad \left(S_1(\mathcal{N}_1) \otimes \bigoplus_{\substack{B\subset\{2,\ldots,d\}\\ \#B=q-1}} \left(\bigotimes_{k=2}^{d} S_{\chi_B(k)}(\mathcal{N}_k) \right) \right) \\
&\cong \big(S_0(\mathcal{N}_1) \otimes S_q(\mathcal{N}_{[2,d]})\big) \oplus \big(S_1(\mathcal{N}_1) \otimes S_{q-1}(\mathcal{N}_{[2,d]})\big) \\
&= (\mathfrak{S}(\mathcal{N}_1) \otimes \mathfrak{S}(\mathcal{N}_{[2,d]}))_q
\end{aligned}
\tag{116}
$$

because $S_i(\mathcal{N}_1) = 0$ for $i \geq 2$. The isomorphism is given as

$$
\phi_q(g_1 \otimes \ldots \otimes g_d) = g_1 \otimes (g_2 \otimes \ldots \otimes g_d) \tag{117}
$$

To show that the isomorphism commutes with the boundary mappings, consider a set $B \subset \{1, \ldots, d\}$ of size q, and an element $g = g_1 \otimes \ldots \otimes g_d \in S_q(\mathcal{N}_{[1,d]})$, where $g_k \in S_{\chi_B(k)}(\mathcal{N}_k)$ for every k. First suppose $1 \in B$, then

$$
\delta'_q(g_1 \otimes (g_2 \otimes \ldots \otimes g_d)) = \delta g_1 \otimes (g_2 \otimes \ldots \otimes g_d) - g_1 \otimes \delta_{q-1}(g_2 \otimes \ldots \otimes g_d) \tag{118}
$$

Let $\overline{B} = B \setminus \{1\}$, then $i_{\overline{B}}(j) = i_B(j+1)$ for all $j = 2, \ldots, q$. Then

$$
\begin{aligned}
\delta'_q \phi_q(g) &= \delta g_1 \otimes (g_2 \otimes \ldots \otimes g_d) \\
&\quad - g_1 \otimes \left(\sum_{j=1}^{q-1} (-1)^{j+1} g_2 \otimes \ldots \otimes \delta g_{i_{\overline{B}}(j)} \otimes \ldots \otimes g_d \right) \\
&= \phi_{q-1}(\delta g_1 \otimes \ldots \otimes g_d) + \sum_{j=2}^{q} (-1)^{j+1} \phi_{q-1}(g_1 \otimes \ldots \otimes \delta g_{i_B(j)} \otimes \ldots \otimes g_d) \\
&= \phi_{q-1} \delta_q(g)
\end{aligned}
\tag{119}
$$

A simpler argument is used for the case $1 \notin B$, then

$$\delta'_q(g_1 \otimes (g_2 \otimes \ldots \otimes g_d)) = g_1 \otimes \delta_{q-1}(g_2 \otimes \ldots \otimes g_d) \tag{120}$$

□

Theorem 3. *Given a tensor product spline mesh $\mathcal{N} = \mathcal{N}_1 \times \ldots \times \mathcal{N}_d$ in $\mathbb{R}^d$, where each $\mathcal{N}_k$ is a univariate spline mesh. Then, for every q,*

$$H_q(\mathfrak{S}(\mathcal{N})) \cong \bigoplus_{\substack{B \subset \{1,\ldots,d\} \\ \#B = q}} \left(\bigotimes_{k=1}^{d} H_{\chi_B(k)}(\mathfrak{S}(\mathcal{N}_k)) \right) \tag{121}$$

Proof. We use induction on d, where the case $d = 1$ is obvious. Assume the theorem holds for $d-1$. Since the isomorphisms in Lemma 16 commutes with the boundary mappings, we have an isomorphism between the homology groups of the two chain complexes, hence

$$\begin{aligned} H_q(\mathfrak{S}(\mathcal{N})) &\cong H_q(\mathfrak{S}(\mathcal{N}_1) \otimes \mathfrak{S}(\mathcal{N}_{[2,d]})) \\ &\cong \left(H_0(\mathfrak{S}(\mathcal{N}_1)) \otimes \bigoplus_{\substack{B \subset \{2,\ldots,d\} \\ \#B = q}} \left(\bigotimes_{k=2}^{d} H_{\chi_B(k)}(\mathfrak{S}(\mathcal{N}_k)) \right) \right) \oplus \\ &\quad \left(H_1(\mathfrak{S}(\mathcal{N}_1)) \otimes \bigoplus_{\substack{B \subset \{2,\ldots,d\} \\ \#B = q-1}} \left(\bigotimes_{k=2}^{d} H_{\chi_B(k)}(\mathfrak{S}(\mathcal{N}_k)) \right) \right) \\ &\cong \bigoplus_{\substack{B \subset \{1,\ldots,d\} \\ \#B = q}} \left(\bigotimes_{k=1}^{d} H_{\chi_B(k)}(\mathfrak{S}(\mathcal{N}_k)) \right) \end{aligned} \tag{122}$$

□

Corollary 2. *Suppose $\mathcal{N} = \mathcal{N}_1 \times \cdots \times \mathcal{N}_d$, where $\mathcal{N}_k = \mathcal{N}(\boldsymbol{a}_k, \boldsymbol{\mu}_k, p_k)$ as in Definition 13. Define*

$$\alpha_k = \sum_{j=1}^{n_k} \mu_{k,j} - (p_k + 1) \tag{123}$$

Let q' be the number of k such that $\alpha_k \geq 0$. Then

$$\dim H_q(\mathfrak{S}(\mathcal{N})) = \begin{cases} | \prod_{k=1}^{d} \alpha_k | & \text{if } q = q' \\ 0 & \text{if } q \neq q' \end{cases} \tag{124}$$

In particular, if $\alpha_k = 0$ for some k, then $\dim H_q(\mathfrak{S}(\mathcal{N})) = 0$ for every q.

Proof. This follows from Theorem 3 and Lemma 15. □

Corollary 3. *Let $\mathcal{N}$ be a tensor product spline mesh such that $\mathbb{S}(\mathcal{N})$ contains non-trivial functions. Then all homology terms in (98) are zero.*

Proof. If $\mathbb{S}(\mathcal{N})$ has non-zero functions, $\alpha_k \geq 1$ for all k. □

Corollary 4. *If the tensor produt spline mesh $\mathcal{N}$ is open, then all homology terms in (99) are zero.*

As we now know the homology for tensor product splines, we can relate this to a simple way to find the 0th homology group in the spline formula for spline meshes in general.

Lemma 17. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$. For each $k = 1, \ldots, d$ and $a \in \mathbb{R}$, let $m_{k,a}$ be the maximum of $\mu(\beta)$ for all (k, a)-meshrectangles $\beta \in \mathcal{M}$, or 0 if no such meshrectangles exists. Define $m_k = p_k + 1 - \sum_{a \in \mathbb{R}} m_{k,a}$. Then*

$$\dim H_0(\mathfrak{S}(\mathcal{N})) = \begin{cases} m_1 m_2 \cdots m_d & \text{if } m_k > 0 \text{ for all } k \\ 0 & \text{if } m_k \leq 0 \text{ for at least one } k \end{cases} \tag{125}$$

In particular, if $\dim H_0(\mathfrak{S}(\mathcal{N})) > 0$, then $\dim \mathbb{S}(\mathcal{N}) = 0$.

Proof. From Lemma 13 we have $H_0(\mathfrak{S}(\mathcal{N})) \cong H_{-1}(\mathfrak{I}(\mathcal{N}))$. The latter is given as

$$H_{-1}(\mathfrak{I}(\mathcal{N})) = \Pi_{\boldsymbol{p}} / \sum_{\beta \in \mathcal{F}_0(\mathcal{M})} \Delta_\beta(\mathcal{N}) \tag{126}$$

The denominator is generated by $(x_k - a)^{p_k + 1 - \mu_k(\beta)}$ taken over all k, a, β such that $\beta \in \mathcal{F}_0(\mathcal{M})$ with $\{a\}$ as its kth component. As every meshrectangle $\gamma \in \mathcal{M}$ also contains a point in $\mathcal{F}_0(\mathcal{M})$ (just start with γ and a point $\boldsymbol{q} \in \partial\gamma$, then $\beta_{\boldsymbol{q}}$ is a box being a proper subset of γ (by Lemma 1), pick a new point on the boundary of $\beta_{\boldsymbol{q}}$ to get a smaller box etc, this eventually leads to a point β in γ), the denominator is also generated by $(x_k - a)^{p_k+1-\mu(\beta)}$ taken over all k, a, β such that β is a (k,a)-meshrectangle.

If we define the tensor product spline mesh $\mathcal{N}' = (\mathcal{M}^T, \mu', \boldsymbol{p})$ where, for each (k,a)-meshrectangle $\beta \in \mathcal{M}^T$, the multiplicity $\mu'(\beta)$ is the maximum of all $\mu(\gamma)$ for all (k,a)-meshrectangles $\gamma \in \mathcal{M}$, then the denominator becomes the same for $\mathcal{N}'$ as for $\mathcal{N}$, and so $H_{-1}(\mathfrak{I}(\mathcal{N})) = H_{-1}(\mathfrak{I}(\mathcal{N}')) \cong H_0(\mathfrak{S}(\mathcal{N}'))$. The latter is given as $\prod_k \max(0, m_k)$ by Corollary 2.

Finally, if $\dim H_{-1}(\mathfrak{I}(\mathcal{N})) > 0$, then each m_k must be positive, then there ar to few knots in each parameter direction to define any non-trivial spline function, and so the spline space $\mathbb{S}(\mathcal{N}')$ only holds the zero function. Then this is also the case for $\mathbb{S}(\mathcal{N})$ because $\mathbb{S}(\mathcal{N}) \subseteq \mathbb{S}(\mathcal{N}')$. □

8 The homology term on tensor expansions

We are going to show that the homologies in the dimensions forumlas are the same for a spline mesh $\mathcal{N}$ and its tensor expansion $\mathcal{N}^T$ in Definition 14. There are a couple of advantages with this. First of all, when investigating a spline mesh, we may always assume it has a tensor grid structure, maybe with zero multiplicity in some meshrectangles. Secondly, inserting new meshrectangles in the mesh or raising the multiplicity in an existing meshrectangle can be treated in the same matter, as meshrectangle insertion is the same as raising the multiplicity from zero in an existing meshrectangle.

To achieve the result, we define a more general chain complex on a homology-suitable box collection related to a spline mesh.

Definition 24. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$ and a box $\beta \subseteq \mathbb{R}^d$. We define the vector space*

$$\Delta_\beta(\mathcal{N}) = \sum_{\substack{\gamma \in \mathcal{M} \\ \beta \subseteq \gamma}} \Delta_\gamma(\mathcal{N}) \tag{127}$$

This generalizes the definition of $\Delta_\beta(\mathcal{N})$ from Definition 20 in case $\beta \in \mathcal{F}(\mathcal{M})$.

If we are also given a homology-suitable box collection $\mathcal{F}$ in $\mathbb{R}^d$, we define the vector space

$$I_q(\mathcal{F}, \mathcal{N}) = \bigoplus_{\beta \in \mathcal{F}_q(\mathcal{M})} [\beta] \Delta_\beta(\mathcal{N}) \tag{128}$$

for any $q = 0, \dots, d$. *Then we define the chain complex*

$$\mathfrak{I}(\mathcal{F}, \mathcal{N}) : 0 \xrightarrow{\delta_{d+1}} I_d(\mathcal{F}, \mathcal{N}) \xrightarrow{\delta_d} I_{d-1}(\mathcal{F}, \mathcal{N}) \xrightarrow{\delta_{d-1}} \dots \xrightarrow{\delta_1} I_0(\mathcal{F}, \mathcal{N}) \xrightarrow{\delta_0} 0 \tag{129}$$

by reusing the δ_q *from the* $\mathfrak{R}(\mathcal{F})$ *chain complex.*

Of course, $\mathfrak{I}(\mathcal{N})$ is the same as $\mathfrak{I}(\mathcal{F}(\mathcal{M}), \mathcal{N})$.

Lemma 18. *Given the homology-suitable box collection* $\mathcal{F}$ *in* $\mathbb{R}^d$ *and the boxes* β, β_1, β_2 *and* γ *from Lemma 11. Also given a spline mesh* $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ *in* $\mathbb{R}^d$. *If every meshrectangle in* $\mathcal{M}$ *that contains* γ *also contains* β, *then*

$$H_q(\mathfrak{I}(\mathcal{F}, \mathcal{N})) \cong H_q(\mathfrak{I}(\mathcal{F}^+, \mathcal{N})) \tag{130}$$

for all q.

Proof. It is the same set of meshrectangles in $\mathcal{M}$ than contain β, β_1, β_2 and γ. Then $\Delta_\beta(\mathcal{N}) = \Delta_{\beta_1}(\mathcal{N}) = \Delta_{\beta_2}(\mathcal{N}) = \Delta_\gamma(\mathcal{N})$. Then the procedure that creates the diagram (61) can be used to get the commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{\ell+1}(\mathcal{F}, \mathcal{N}) & \xrightarrow{\psi_{\ell+1}} & I_{\ell+1}(\mathcal{F}^+, \mathcal{N}) & \rightarrow & 0 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{\ell}(\mathcal{F}, \mathcal{N}) & \xrightarrow{\psi_{\ell}} & I_{\ell}(\mathcal{F}^+, \mathcal{N}) & \rightarrow & [\beta_1]\Delta_\beta(\mathcal{N}) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\psi & & \\
0 & \rightarrow & I_{\ell-1}(\mathcal{F}, \mathcal{N}) & \xrightarrow{\psi_{\ell-1}} & I_{\ell-1}(\mathcal{F}^+, \mathcal{N}) & \rightarrow & [\gamma]\Delta_\beta(\mathcal{N}) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{\ell-2}(\mathcal{F}, \mathcal{N}) & \xrightarrow{\psi_{\ell-2}} & I_{\ell-2}(\mathcal{F}^+, \mathcal{N}) & \rightarrow & 0 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{131}$$

with complexes in the columns and exact sequences on the rows. The long exact sequence of homologies yields the result. □

Theorem 4. *For a spline mesh* $\mathcal{N}$, $H_q(\mathfrak{I}(\mathcal{N})) \cong H_q(\mathfrak{I}(\mathcal{N}^T))$ *for all* q.

Proof. We transform the box collection $\mathcal{F}(\mathcal{M})$ into $\mathcal{F}(\mathcal{M}^T)$ by a series of modifications as in Lemma 11. Notice that for any box collection $\mathcal{F}$ during this process, we always have the following property: For any box $\alpha \in \mathcal{F}$, any point $\boldsymbol{q} \in \alpha \setminus \partial\alpha$ and any box $\alpha' \in \mathcal{F}(\mathcal{M})$ such that $\boldsymbol{q} \in \alpha'$, we must have $\alpha \subseteq \alpha'$. To prove this, we start with the initial box collection $\mathcal{F} = \mathcal{F}(\mathcal{M})$. Then we have $\alpha = \beta_{\boldsymbol{q}}$ by Lemma 1. It is the intersection of all boxes in $\mathcal{E}(\mathcal{F})$ containing $\boldsymbol{q}$, while α' is the intersection of some of these boxes. Therefore $\alpha \subseteq \alpha'$. Then we move on by induction and check the transformation from $\mathcal{F}$ to $\mathcal{F}^+$. In that case, the box β is replaced by three boxes β_1, β_2 and γ. As all of their interior points also are interior points of β, the property is conserved.

Now look at the step from $\mathcal{F}$ to $\mathcal{F}^+$, and let α be a meshrectangle in $\mathcal{M}$ containing γ. Then α also contains a point $\boldsymbol{q}$ in $\gamma \setminus \partial\gamma$. This is also a point in $\beta \setminus \partial\beta$ and therefore, by the property just proven, we must have $\beta \subseteq \alpha$. So every meshrectangle in $\mathcal{M}$ containing γ will also contain β, then we can use Lemma 18 to get $H_q(\mathfrak{I}(\mathcal{F},\mathcal{N})) \cong H_q(\mathfrak{I}(\mathcal{F}^+,\mathcal{N}))$ for all q. Going through all the transformation steps yields $H_q(\mathfrak{I}(\mathcal{N})) \cong H_q(\mathfrak{I}(\mathcal{F}(\mathcal{M}^T),\mathcal{N}))$ for all q.

The next step is to prove $H_q(\mathfrak{I}(\mathcal{F}(\mathcal{M}^T),\mathcal{N})) = H_q(\mathfrak{I}(\mathcal{N}^T))$ for all q. The chain complexes for these homologies are based on the same set of boxes, $\mathcal{F}(\mathcal{M}^T)$, so it is sufficient to show $\Delta_\beta(\mathcal{N}^T) = \Delta_\beta(\mathcal{N})$ for every $\beta \in \mathcal{F}(\mathcal{M}^T)$. As $\Delta_\beta(\mathcal{N})$ is defined by the p_k and $\mu_k(\beta)$, we only need to show that $\mu_k(\beta) = \mu_k^T(\beta)$ for all k and all $\beta \in \mathcal{F}(\mathcal{M}^T)$.

Given a k-meshrectangle $\gamma \in \mathcal{D}_\beta^k(\mathcal{M}^+)$. If $\gamma \subseteq \gamma'$ for some k-meshrectangle $\gamma' \in \mathcal{M}$, then $\mu^T(\gamma) = \mu(\gamma')$ and $\gamma' \in \mathcal{D}_\beta^k(\mathcal{M})$, if not, then $\mu^T(\gamma) = 0$. Using this for every $\gamma \in \mathcal{D}_\beta^k(\mathcal{M}^+)$ gives $\mu_k^T(\beta) \leq \mu_k(\beta)$. On the other hand, given a meshrectangle $\gamma \in \mathcal{D}_\beta^k(\mathcal{M})$, in particular it contains a point $\boldsymbol{q} \in \beta \setminus \partial\beta$. During the process from $\mathcal{M}$ to $\mathcal{M}^T$, γ might be split several times into smaller k-meshrectangles, at least one of them, γ', will contain $\boldsymbol{q}$. As $\beta = \beta_{\boldsymbol{q}}$ by Lemma 1, we must have $\beta \subseteq \gamma'$. Then $\gamma' \in \mathcal{D}_\beta^k(\mathcal{M}^+)$ and $\mu(\gamma) = \mu^T(\gamma')$ giving $\mu_k(\beta) \leq \mu_k^T(\beta)$. Altogether we have $\mu_k(\beta) = \mu_k^T(\beta)$, completing the proof. □

9 Changes in the homologies under subdivision

In this section we take a look at how homologies are affected during the insertion of new meshrectangles as well as multiplicity elevation of existing ones. These two cases are treated similarly: In either case we extend the spline mesh $\mathcal{N}$ to $\mathcal{N}^T$ (maybe with an extra knot insertion where all new meshrectangles are given mulitplicity 0), then meshrectangle insertion is the same as raising the multiplicity from 0 to 1.

Because we can move to $\mathcal{N}^T$ without changing the homologies, we always work over a tensor product spline mesh.

9.1 The mesh extension $\mathcal{N} + B$

Definition 25. *Given a boxmesh $\mathcal{M}$ in $\mathbb{R}^d$, an integer κ and a number $a \in \mathbb{R}$. A set $B \subseteq \mathcal{M}$ is a (κ, a)-**meshcollection** of $\mathcal{M}$ if every $\beta \in B$ is a (κ, a)-meshrectangle. A set B is also called a **meshcollection** or a κ-**meshcollection** of $\mathcal{M}$ if it is a (κ, a)-meshcollection of $\mathcal{M}$ for som κ and a. If B is a meshcollection, we define $\bar{B} = \cup_{\beta \in B} \beta$. A **meshblock** or κ-**meshblock** or (κ, a)-**meshblock** of $\mathcal{M}$ is a meshcollection B of $\mathcal{M}$ such that $\bar{B}$ is a meshrectangle (respectively κ-meshrectangle or (κ, a)-meshrectangle).*

*Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$. A meshcollection B of $\mathcal{M}$ has a **uniform multiplicity** in $\mathcal{N}$ if all meshrectangles $\beta \in B$ have the same multiplicity, i.e. if $\mu(\beta) = \nu$ for some ν for all $\beta \in B$. The number ν is the **multiplicity of** B **in** $\mathcal{N}$ and is denoted $\mu(B)$. If B is a κ-meshcollection and m is an integer, B has m-**proper uniform multiplicity** in $\mathcal{N}$ if $\mu(B) \leq p_k + 1 - m$, and B has **proper uniform multiplicity** in $\mathcal{N}$ if it has 1-proper uniform multiplicity in $\mathcal{N}$.*

Definition 26. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$, a number $m \geq 0$ and a κ-meshcollection B of $\mathcal{M}$ such that $\mu(\beta) \leq p_\kappa + 1 - m$ for all $\beta \in B$ (a special case is when B is of m-proper uniform multiplicity). We then define $\mathcal{N} + mB$ to be the spline mesh $(\mathcal{M}, \mu^+, \boldsymbol{p})$ in $\mathbb{R}^d$ where*

$$\mu^+(\beta) = \mu(\beta) + m\chi_B(\beta) = \begin{cases} \mu(\beta) & \text{if } \beta \notin B \\ \mu(\beta) + m & \text{if } \beta \in B \end{cases} \tag{132}$$

For the case $m = 1$ we write $\mathcal{N} + B$ for $\mathcal{N} + 1B$.

We have some obvious properties

- $(\mathcal{N} + m_1 B_1) + m_2 B_2 = (\mathcal{N} + m_2 B_2) + m_1 B_1$
- $\mathcal{N} + 0B = \mathcal{N}$
- $\mathcal{N} + m\emptyset = \mathcal{N}$

- $\mathcal{N} + (m_1 + m_2)B = (\mathcal{N} + m_1 B) + m_2 B$
- $(\mathcal{N} + mB_1) + mB_2 = \mathcal{N} + m(B_1 \cup B_2)$ if B_1 and B_2 are disjoint

9.2 The mesh restriction $\mathcal{N} \cap B$

The next step is to look at how we can define a spline mesh on a meshbox B by restricting it from $\mathcal{N}$. We start with some definitions.

Definition 27. *Given an $(\ell, d-1)$-box*

$$\beta = J_1 \times \cdots \times J_{d-1} , \tag{133}$$

a closed interval or point $J \subseteq \mathbb{R}$ and an integer $\kappa \in \{1, \ldots, d\}$. Then we define the box $\theta(\beta, J, \kappa)$ in $\mathbb{R}^d$ by

$$\theta(\beta, J, \kappa) = J_1 \times \cdots \times J_{\kappa-1} \times J \times J_\kappa \times \cdots \times J_{d-1} \tag{134}$$

If J is a point, $\theta(\beta, J, \kappa)$ is an (ℓ, d)-box, and if J is a closed interval, $\theta(\beta, J, \kappa)$ is an $(\ell+1, d)$-box.

We also define $s(k, \kappa)$ to be k if $k < \kappa$ and $k+1$ if $k \geq \kappa$. Thus, the component J_k becomes component number $s(k, \kappa)$ in $\theta(\beta, J, \kappa)$.

If $\mathcal{M} = \mathcal{M}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ is a tensor mesh, where $\boldsymbol{a}_k = (a_{k,1}, \ldots, a_{k,n_k})$ is an increasing sequence of length ≥ 2 for all k, and B is a κ-meshblock of $\mathcal{M}$, then the meshrectangle $\bar{B}$ is on the form

$$\bar{B} = [a_{1,e_1}, a_{1,f_1}] \times \cdots \times [a_{d,e_d}, a_{d,f_d}] \tag{135}$$

such that $1 \leq e_k \leq f_k \leq n_k$ for every $k = 1, \ldots, d$ where $e_k = f_k$ if and only if $k = \kappa$. The meshblock B is then given as the set of all meshrectangles β such that the kth component is $\{a_{\kappa,e_\kappa}\}$ if $k = \kappa$ and $[a_{k,j}, a_{k,j+1}]$ for some j such that $e_k \leq j < f_k$ if $k \neq \kappa$.

Definition 28. *Given a spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$, $d \geq 2$ such that $\mathcal{M} = \mathcal{M}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ is a tensor mesh, where $\boldsymbol{a}_k = (a_{k,1}, \ldots, a_{k,n_k})$ is an increasing sequence of length ≥ 2 for all k. Also given $\kappa \in \{1, \ldots, d\}$ and a κ-meshblock B of $\mathcal{M}$ on the form* (135) *such that B has uniform multiplicity in $\mathcal{N}$. We then define $\mathcal{N} \cap B$ to be the spline mesh $(\bar{\mathcal{M}}, \bar{\mu}, \bar{\boldsymbol{p}})$ in $\mathbb{R}^{d-1}$ where*

- $\bar{\mathcal{M}} = \mathcal{M}(\boldsymbol{b}_1, \dots, \boldsymbol{b}_{d-1})$ *is a tensor mesh where*

$$\boldsymbol{b}_k = (a_{s(k,\kappa),e_{s(k,\kappa)}}, a_{s(k,\kappa),e_{s(k,\kappa)}+1}, \dots, a_{s(k,\kappa),f_{s(k,\kappa)}}) \tag{136}$$

for every $k = 1, \dots, d-1$.

- *For any k-meshrectangle* $\beta \in \bar{\mathcal{M}}$,

$$\bar{\mu}(\beta) = \begin{cases} \mu_{s(k,\kappa)}(\theta(\beta, \{e_\kappa\}, \kappa)) & \text{if } \mu_\kappa(\theta(\beta, \{e_\kappa\}, \kappa)) \leq \mu(B) \\ p_{s(k,\kappa)} + 1 & \text{if } \mu_\kappa(\theta(\beta, \{e_\kappa\}, \kappa)) > \mu(B) \end{cases} \tag{137}$$

- $\bar{\boldsymbol{p}} = (p_1, \dots, p_{\kappa-1}, p_{\kappa+1}, \dots, p_d)$

9.3 The homology relation between $\mathcal{N}$, $\mathcal{N}+B$ and $\mathcal{N} \cap B$

The main result in this section is the relation between the homologies for $\mathcal{N}$, $\mathcal{N}+B$ and $\mathcal{N} \cap B$ in the case when B fits well into $\mathcal{N}$, which is described by the following definition.

Definition 29. *Given a spline mesh* $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ *in* $\mathbb{R}^d$, $d \geq 2$ *and a* κ*-meshblock* B *of* $\mathcal{M}$ *of uniform multiplicity* $\mu(B)$. *We say that* B **slots into** $\mathcal{N}$ *if for every* $\beta \in \mathcal{F}(\mathcal{M})$ *and for every k-meshrectangle* $\gamma \in \mathcal{M}$ *such that* $\beta \subseteq \gamma \cap \bar{B}$ *and* $\mu(\gamma) > \mu(B)$ *if* $k = \kappa$, *there exists a k-meshrectangle* $\gamma' \in \mathcal{M}$ *such that* $\beta \subseteq \gamma' \cap \bar{B}$, $\dim(\gamma' \cap \bar{B}) = d - 2$ *and*

- $\mu(\gamma') \geq \mu(\gamma)$ *if* $k \neq \kappa$
- $\mu(\gamma') > \mu(B)$ *if* $k = \kappa$

Notice that it is sufficient to expect $\dim(\gamma' \cap \bar{B}) \geq d - 2$ as the dimension can not be more than $d-2$. Either $k \neq \kappa$, then $\bar{B}$ and γ' are meshrectangles of different parameter directions, or $k = \kappa$, then the interior of γ' lies outside $\bar{B}$ because $\mu(\gamma') \neq \mu(B)$.

It follows from the definition that for a spline mesh $\mathcal{N}$ in $\mathbb{R}^2$, a κ-meshblock of uniform multiplicity always slots into $\mathcal{N}$. This is because we can choose $\gamma' = \gamma$, then $\dim(\gamma' \cap \bar{B}) \geq \dim \beta \geq 0$.

An example of how meshblocks do and do not slot into a mesh is shown in Figure 5. All shown inner meshrectangles β have $\mu(\beta) = 1$. To the left a spline mesh $\mathcal{N}$ with a previously inserted block C before we insert the next mesh block. Picture 2 shows how a new meshblock B does not slot into $\mathcal{N}$.

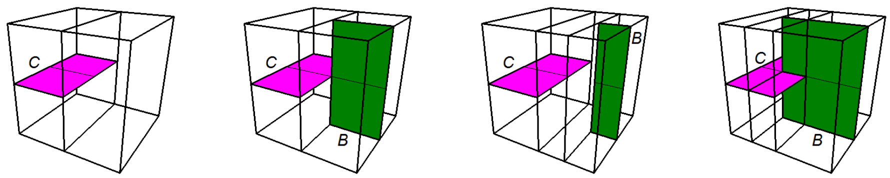


Figure 5: To the left a mesh in $\mathbb{R}^3$ before a new insertion. Then an example of a new meshblock that does not slot into the mesh, followed by two examples where the meshblock does slot into the mesh. See text for details.

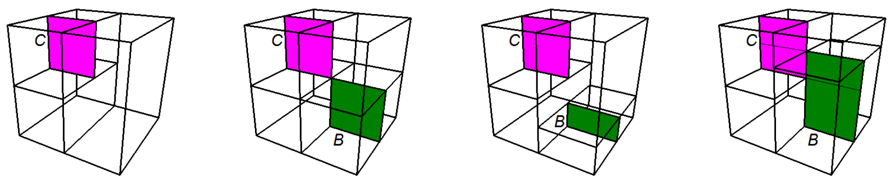


Figure 6: Similar to Figure 5, but this time C and B are k-meshblocks for the same k.

This is because for the boint $\beta = \bar{C} \cap \bar{B}$, we can find a meshrectangle $\gamma \in C$ containing β and with $\mu(\gamma) = 1$, but there is no meshrectangle γ' parallell to C, containing β such that $\mu(\gamma') \geq 1$ and $\dim(\gamma' \cap \bar{B}) = 1$. On picture 3, the new meshblock slots into $\mathcal{N}$ because C causes no problem this time since $\bar{C} \cap \bar{B}$ is empty. In the last picture, $\bar{C} \cap \bar{B}$ is a line, this will also have the consequence that B slots into $\mathcal{N}$. We see in the picture that B slots into C like cardboards or other thin surfaces of material somtimes are slot together to make stable physical items.

Another example is shown in Figure 6, this time the meshblocks are parallell. Just like in the previous example, Picture 1 shows the mesh before insertion, Picture 2 shows how B does not slot into $\mathcal{N}$ because $\bar{C} \cap \bar{B}$ is a point, while Picture 3 and 4 show meshblocks that slot into $\mathcal{N}$ because $\bar{C} \cap \bar{B}$ is either empty or of dimension 1.

Theorem 5. *Given a tensor product spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$, $d \geq 2$, and a meshblock B of $\mathcal{M}$ of proper uniform multiplicity. If B slots into $\mathcal{N}$, then there is an exact sequence*

$$0 \to I_q(\mathcal{N}) \to I_q(\mathcal{N} + B) \to S_q(\mathcal{N} \cap B) \to 0 \tag{138}$$

for $q = -1, \ldots, d-1$ (using $S_{-1}(\mathcal{N} \cap B) = 0$), and the maps in these sequences commute with the boundary mappings δ_q in the chain complexes $\widetilde{\mathfrak{I}}(\mathcal{N})$, $\widetilde{\mathfrak{I}}(\mathcal{N} + B)$ and $\mathfrak{S}(\mathcal{N} \cap B)$.

Proof. We assume B is a κ-meshblock. For $q = -1$ we use the exact sequence $0 \to \Pi_{\boldsymbol{p}} = \Pi_{\boldsymbol{p}} \to 0 \to 0$. Therefore we continue with $q \geq 0$. Let $\mathcal{M} = \mathcal{M}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d)$ with the $\boldsymbol{a}_k$ described as in Definition 28. For any meshrectangle $\gamma \in \mathcal{M}$ we have $\mu(\gamma) \leq \mu^+(\gamma)$, then for any box $\beta \in \mathcal{F}(\mathcal{M})$ and any $k = 1, \ldots, d$ we get $\mu_k(\beta) \leq \mu_k^+(\beta)$. For any point $\boldsymbol{q} \in \beta$, the vector spaces $\Delta_\beta(\mathcal{N})$ and $\Delta_\beta(\mathcal{N} + B)$ are generated by factors of the polynomials $(x_k - q_k)^{p_k - \mu_k(\beta)+1}$ and $(x_k - q_k)^{p_k - \mu_k^+(\beta)+1}$ respectively, hence $\Delta_\beta(\mathcal{N}) \subseteq \Delta_\beta(\mathcal{N} + B)$. Combining this for all q-boxes yields an exact sequence

$$0 \to I_q(\mathcal{N}) \to I_q(\mathcal{N} + B) \to \bigoplus_{\beta \in \mathcal{F}_q(\mathcal{M})} [\beta] V_\beta \to 0 \tag{139}$$

where

$$V_\beta = \Delta_\beta(\mathcal{N} + B) / \Delta_\beta(\mathcal{N}) \tag{140}$$

If $\beta \notin B$, then $\gamma \nsubseteq \bar{B}$ for any $\gamma \in \mathcal{D}_\beta(\mathcal{M})$. Then $\mu(\gamma) = \mu^+(\gamma)$, hence $\mu_k(\beta) = \mu_k^+(\beta)$ for all k, and therefore $V_\beta = 0$. So the free sum in (139) should only be over boxes $\beta \in B$.

There is a one-to-one correspondance between the boxes $\beta \in \mathcal{F}_q(\mathcal{M})$ such that $\beta \notin B$ and the boxes $\beta' \in \mathcal{F}_q(\bar{\mathcal{M}})$ given by $\beta = \theta(\beta', \{e_\kappa\}, \kappa)$. Fix β and the corresponding β', and let $\boldsymbol{q}' = (q'_1, \ldots, q'_{d-1})$ be a point on β'. Clearly there are κ-meshrectangles in B containing β, therefore we have $\mu_\kappa(\beta) \geq \mu(B)$. We want to show

$$V_\beta \cong \Pi_{\bar{\boldsymbol{p}}} / \Delta_{\beta'}(\mathcal{N} \cap B) \tag{141}$$

and split this proof into two cases.

The first case is when $\mu_\kappa(\beta) > \mu(B)$. Then there is a $\gamma \in \mathcal{D}_\beta^\kappa(\mathcal{M})$ such that $\mu(\gamma) > \mu(B)$. Then also $\mu^+(\gamma) = \mu(\gamma)$ because we must have $\gamma \notin B$, and so $\mu_\kappa^+(\beta) = \mu_\kappa(\beta)$ hence $V_\beta = 0$. Because B slots into $\mathcal{N}$, we can find a $\gamma' \in \mathcal{D}_\beta^\kappa(\mathcal{M})$ such that $\mu(\gamma') > \mu(B)$ and $\dim \gamma' \cap \bar{B} = d-2$. If we remove component number κ from $\gamma' \cap \bar{B}$, we get a box $\gamma'' \in \mathcal{F}(\bar{\mathcal{M}})$ of dimension $d-2$. Because of the the tensor product structure of $\mathcal{M}$, γ'' is a k-meshrectangle in $\bar{\mathcal{M}}$ for some k, and we also have $\gamma' \cap \bar{B} = \theta(\gamma'', \{e_\kappa\}, \kappa)$. Then $\mu_\kappa(\theta(\gamma'', \{e_\kappa\}) \geq \mu(\gamma') > \mu(B)$ and so by definition $\bar{\mu}_k(\beta) = p_{s(k,\kappa)}+1 =$

$(\bar{\boldsymbol{p}})_k$. From this we get $\bar{\mu}_k(\beta') = p_{s(k,\kappa)} + 1$. The vector space $\Delta_{\beta'}(\mathcal{N} \cap B)$ therefore contains multiples of $(y_k - q'_k)^{(\bar{\boldsymbol{p}})_k - \bar{\mu}_k(\beta)+1} = (y_k - q'_k)^0 = 1$ (we use y_k as unknowns in $\Pi_{\bar{\boldsymbol{p}}}$ and x_k as unknowns in $\Pi_{\boldsymbol{p}}$). Then $\Delta_{\beta'}(\mathcal{N} \cap B) = \Pi_{\bar{\boldsymbol{p}}}$, hence $\Pi_{\bar{\boldsymbol{p}}}/\Delta_{\beta'}(\mathcal{N} \cap B) = 0 = V_\beta$.

The second case is for $\mu_\kappa(\beta) = \mu(B)$. This time we have $\mu^+_{\boldsymbol{\kappa}}(\beta) = \mu(B)+1$. A basis for $\Pi_{\boldsymbol{p}}$ is given by the polynomials

$$\left\{\prod_{k=1}^{d}(x_k - q_k)^{i_k}, 0 \le i_k \le p_k \forall k\right\} \tag{142}$$

for the point $\boldsymbol{q} = (q_1, \ldots, q_d) = \theta(\boldsymbol{q}', \{e_\kappa\}, \kappa)$. The space $\Delta_\beta(\mathcal{N})$ is spanned by the polynomials in (142) where $i_k \ge p_k - \mu_k(\beta) + 1$ for at least one k, and $\Delta_\beta(\mathcal{N} + B)$ is spanned by the polynomials where $i_k \ge p_k - \mu_k(\beta) + 1$ for at least one $k \ne \kappa$, or where $i_\kappa \ge p_\kappa - \mu_\kappa(\beta)$. Then we have a natural basis for $\Delta_\beta(\mathcal{N} + B)/\Delta_\beta(\mathcal{N})$ given by the polyonmials in (142) where $i_k \le p_k - \mu_k(\beta)$ for all $k \ne \kappa$ and where $i_\kappa = p_\kappa - \mu(B)$.

On the other hand, a basis for $\Pi_{\bar{\boldsymbol{p}}}$ is given by the polynomials

$$\left\{\prod_{k=1}^{d-1}(y_k - q'_k)^{j_k}, 0 \le j_k \le p_{s(k,\kappa)} \forall k\right\} \tag{143}$$

The space $\Delta_{\beta'}(\mathcal{N} \cap B)$ is generated by the polynomials in (143) where $j_k \ge p_{s(k,\kappa)} - \bar{\mu}_k(\beta') + 1$ for at least one k. Then we have a natural basis for $\Pi_{\bar{\boldsymbol{p}}}/\Delta_{\beta'}(\mathcal{N} \cap B)$ given by the polyonmials in (142) where $j_k \le p_{s(k,\kappa)} - \bar{\mu}_k(\beta')$ for all $k = 1, \ldots, d-1$. Therefore, the linear map

$$\begin{aligned} \xi : \Pi_{\bar{\boldsymbol{p}}} &\to \Pi_{\boldsymbol{p}} \\ g(y_1, \ldots, y_{d-1}) &\mapsto g(x_{s(1,\kappa)}, \ldots, x_{s(d-1,\kappa)})(x_\kappa - e_\kappa)^{p_\kappa - \mu(B)} \end{aligned} \tag{144}$$

induces an isomorphism

$$\bar{\xi} : \Pi_{\bar{\boldsymbol{p}}}/\Delta_{\beta'}(\mathcal{N} \cap B) \to V_\beta \tag{145}$$

if we can show that $\bar{\mu}_k(\beta') = \mu_{s(k,\kappa)}(\beta)$ for all $k = 1, \ldots, d-1$. We do this by showing two inequalities.

First given $\gamma' \in \mathcal{D}^k_{\beta'}(\bar{\mathcal{M}})$, set $\gamma = \theta(\gamma', \{e_\kappa\}, \kappa)$. We have $\beta \subseteq \gamma$ because $\beta' \subseteq \gamma'$. This gives $\mu_\kappa(\gamma) \le \mu_\kappa(\beta) = \mu(B)$, by definition we then have

$$\bar{\mu}(\gamma') = \mu_{s(k,\kappa)}(\gamma) \le \mu_{s(k,\kappa)}(\beta)\ . \tag{146}$$

Take the maximum over all γ' to get $\bar{\mu}_k(\beta') \leq \mu_{s(k,\kappa)}(\beta)$.

Then given $\gamma \in \mathcal{D}_\beta^{s(k,\kappa)}(\mathcal{M})$. Because B slots into $\mathcal{N}$, there exists $\gamma \in \mathcal{D}_\beta^{s(k,\kappa)}(\mathcal{M})$ such that $\dim(\gamma' \cap \bar{B}) = d-2$ and $\mu(\gamma) \leq \mu(\gamma')$. We have $\gamma' \cap \bar{B} = \theta(\gamma'', \{e_\kappa\}, \kappa)$ for some $\gamma'' \in \mathcal{D}_{\beta'}^k(\bar{\mathcal{M}})$. By definition we have $\mu_{s(k,\kappa)}(\gamma' \cap B) \leq \bar{\mu}(\gamma'')$. Altogether we get

$$\mu(\gamma) \leq \mu(\gamma') \leq \mu_{s(k,\kappa)}(\gamma' \cap B) \leq \bar{\mu}(\gamma'') \leq \bar{\mu}_k(\beta') \tag{147}$$

Take the maximum over all γ to get $\mu_{s(k,\kappa)}(\beta) \leq \bar{\mu}_k(\beta')$. This completes the proof that $\bar{\mu}_k(\beta') = \mu_{s(k,\kappa)}(\beta)$, and also that (141) holds for all $\beta \in B$. Then

$$\bigoplus_{\beta \in \mathcal{F}_q(\mathcal{M})} [\beta] V_\beta \cong \bigoplus_{\beta' \in \mathcal{F}_q(\bar{\mathcal{M}})} [\beta'] \Pi_{\bar{\boldsymbol{p}}} / \Delta_{\beta'}(\mathcal{N} \cap B) = \mathfrak{S}(\mathcal{N} \cap B) \tag{148}$$

and so the exact sequence (138) is established from (139). For any box $\beta \in B$, component κ is trivial, so if the non-trivial component number j of $\beta' \in \mathcal{F}(\bar{\mathcal{M}})$ is in parameter direction k, then the non-trivial component number j of $\theta(\beta', \{e_\kappa\}, \kappa)$ is in parameter direction $s(k, \kappa)$. Therefore the maps δ_q commute with the exact sequences. □

Corollary 5. *Given a tensor product spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$, $d \geq 2$, and a meshblock B of $\mathcal{M}$ of proper uniform multiplicity. If B slots into $\mathcal{N}$, then there is an exact sequence*

$$\begin{aligned} 0 &\to H_d(\mathfrak{S}(\mathcal{N})) \to H_d(\mathfrak{S}(\mathcal{N}+B)) \to H_{d-1}(\mathfrak{S}(\mathcal{N} \cap B)) \\ &\to H_{d-1}(\mathfrak{S}(\mathcal{N})) \to H_{d-1}(\mathfrak{S}(\mathcal{N}+B)) \to H_{d-2}(\mathfrak{S}(\mathcal{N} \cap B)) \\ &\to \cdots \to H_0(\mathfrak{S}(\mathcal{N})) \to H_0(\mathfrak{S}(\mathcal{N}+B)) \to 0 \end{aligned} \tag{149}$$

Proof. Use Lemma 5 and Lemma 13. □

Corollary 6. *Given a tensor product spline mesh $\mathcal{N} = (\mathcal{M}, \mu, \boldsymbol{p})$ in $\mathbb{R}^d$, $d \geq 2$, and a meshblock B of $\mathcal{M}$ of proper uniform multiplicity such that B slots into $\mathcal{N}$. Then, for any $q = 0, \ldots, d$*

$$\dim H_q(\mathfrak{S}(\mathcal{N}+B)) \leq \dim H_q(\mathfrak{S}(\mathcal{N})) + \dim H_{q-1}(\mathfrak{S}(\mathcal{N} \cap B)) \tag{150}$$

Proof. The sequence

$$H_q(\mathfrak{S}(\mathcal{N})) \to H_q(\mathfrak{S}(\mathcal{N}+B)) \to H_{q-1}(\mathfrak{S}(\mathcal{N} \cap B)) \tag{151}$$

is exact in the middle □

To arrive at a spline mesh, we can always start with an ordinary tensor product spline mesh and do a sequence of multiplicity elevations $\mathcal{N} \to \mathcal{N} + B$ of boxmeshes B. Corollary 6 gives a way to get an upper limit of the homology term in the dimension formula in case B slots into $\mathcal{N}$ for every such multiplicity elevation step. If the starting tensor product spline mesh is open, then so is the resulting mesh.

Theorem 6. *Let $\mathcal{N}_0$ be a tensor product spline mesh in $\mathbb{R}^d$ and let $B_1, \dots, B_n$ be a sequence of boxmeshes in $\mathcal{N}$. Define $\mathcal{N}_i = \mathcal{N}_{i-1} + B_i$ for $i = 1, \dots, n$ and assume B_i has proper uniform multiplicity in $\mathcal{N}_{i-1}$ for every i. Let $\mathcal{N} = \mathcal{N}_n$. If B_i slots into $\mathcal{N}_{i-1}$ for every i, then*

$$\dim H_q(\mathfrak{S}(\mathcal{N})) \leq \dim H_q(\mathfrak{S}(\mathcal{N}_0)) + \sum_{i=1}^{n} \dim H_{q-1}(\mathfrak{S}(\mathcal{N}_{i-1} \cap B_i)) \tag{152}$$

for every $q = 0, \dots, d-1$.

Proof. We use induction on n, the case $n = 0$ is obvious. For the induction step, we use Corollary 6 to get

$$\begin{aligned} \dim H_q(\mathfrak{S}(\mathcal{N}_n)) &\leq \dim H_q(\mathfrak{S}(\mathcal{N}_{n-1})) + \dim H_{q-1}(\mathfrak{S}(\mathcal{N}_{n-1} \cap B_n)) \\ &\leq \dim H_q(\mathfrak{S}(\mathcal{N}_0)) + \sum_{i=1}^{n} \dim H_{q-1}(\mathfrak{S}(\mathcal{N}_{i-1} \cap B_i)) \end{aligned} \tag{153}$$

□

For the case of an open spline mesh, we can omit the first term

Corollary 7. *Given $\mathcal{N}_0$ and the B_i as in Theorem 6. If $\mathcal{N}_0$ is open, then so is every $\mathcal{N}$, and we have*

$$\dim H_q(\mathfrak{S}^0(\mathcal{N})) = \dim H_q(\mathfrak{S}(\mathcal{N})) \leq \sum_{i=1}^{n} \dim H_{q-1}(\mathfrak{S}(\mathcal{N}_{i-1} \cap B_i)) \tag{154}$$

for every $q = 1, \dots, d-1$.

Proof. Because $\mathcal{N}_0$ is open, we see that $\mathcal{N}_i$ is open for all i, therefore $\mathcal{N}$ is open and $H_q(\mathfrak{S}^0(\mathcal{N}) \cong H_q(\mathfrak{S}(\mathcal{N})$. Furthermore, $H_q(\mathfrak{S}(\mathcal{N}_0)) = 0$ by Corollary 4. □

This result might help us to find upper limits for the homology terms from our d-dimensional spline mesh, by looking into homlogy terms in spline meshes of dimension $d-1$. What we will give now is a way to construct $\mathcal{N}_{n-1} \cap B_n$ from knowing $\mathcal{N}_0$ and $B_1, \ldots, B_{n-1}$ by subsequent subdivisions on the same $(d-1)$-dimensional boxmesh.

Definition 30. *Let $\mathcal{N}_0$ be an open tensor product spline mesh in $\mathbb{R}^d$ and let $B_1, \ldots, B_n$ be a sequence of boxmeshes in $\mathcal{N}$ where each B_i is a (κ_i, a_i)-meshbox. Define $\mathcal{N}_i = \mathcal{N}_{i-1} + B_i$ for $i = 1, \ldots, n$ and assume B_i has proper uniform multiplicity in $\mathcal{N}_{i-1}$ for every i.*

Define $\mathcal{N}'_{-1} = \mathcal{N}_0 \cap B_n = (\bar{\mathcal{M}}, \bar{\mu}, \bar{\boldsymbol{p}})$. It is a tensor product spline mesh in $\mathbb{R}^{d-1}$ with multiplicities inherited from $\mathcal{N}_0$ in the sense that every (k,b)-meshrectangle in $\mathcal{N}'_{-1}$ has the same multiplicity as every $(s(k,\kappa_n), b)$-meshrectangle in $\mathcal{N}_0$. The boxmesh $\bar{M}$, it is the same as the boxmesh of $\mathcal{N}_{n-1} \cap B_n$.

Let μ^i be the multiplicity function of $\mathcal{N}_i$, i.e. $\mathcal{N}_i = (\mathcal{M}, \mu^i, \boldsymbol{p})$. We define the set $D \subseteq \bar{\mathcal{M}}$ to be all meshrectangles $\beta \in \bar{\mathcal{M}}$ such that if β is a k-meshrectangle, then $\mu^{n-1}_{\kappa_n}(\theta(\beta, \{a_n\}, \kappa_n)) > \mu^{n-1}(B)$. We define $\mathcal{N}'_0$ to be the mesh $(\bar{\mathcal{M}}, \bar{\mu}^+, \bar{\boldsymbol{p}})$ where $\bar{\mu}^+$ is given as

$$\bar{\mu}^+(\beta) = \begin{cases} \bar{\mu}(\beta) & \text{if } \beta \notin D \\ p_{s(k,\kappa_n)} + 1 & \text{if } \beta \in D \end{cases} \tag{155}$$

for any k-meshrectangle $\beta \in \bar{\mathcal{M}}$.

Next, for any $i = 1, \ldots, n-1$, we define the meshcollection $C_i \subset \bar{M}$ and the spline mesh $\mathcal{N}'_i$ as follows.

- *If $\kappa_n = \kappa_i$ then $C_i = \emptyset$*
- *If $\kappa_n \neq \kappa_i$ then C_i is the set of all meshrectangles $\beta \in \bar{\mathcal{M}}$ such that there exists some $\gamma \in B_i$ where $\theta(\beta, \{a_n\}, \kappa_n) \subseteq \gamma$ and $\mu^{n-1}_{\kappa_i}(\theta(\beta, \{a_n\}, \kappa_n)) = \mu^{n-1}(\gamma)$.*
- *$\mathcal{N}'_i = \mathcal{N}'_{i-1} + C_i$*

Lemma 19. *With the construction in the above definition, we have $\mathcal{N}'_{n-1} = \mathcal{N}_{n-1} \cap B_n$.*

Proof. Comes later □

Notice that the meshcollections C_i are not necessarily meshboxes, so in order to get from $\mathcal{N}'_{i-1}$ to $\mathcal{N}_i$, we might have to split into several subdivisions. Also the step from $\mathcal{N}'_{-1}$ to $\mathcal{N}'_0$ might need several steps, on the other hand, if $\mu^{n-1}(\beta) \leq \mu^{n-1}(B_n)$ for every (κ_n, a_n)-meshrectangle $\beta \in \mathcal{M}$, the set D is empty, and $\mathcal{N}'_{-1} = \mathcal{N}'_0$. Also, if the union of all meshrectangles in D is a union of boundary faces in $\Omega(\mathcal{M})$, the spline mesh $\mathcal{N}'_0$ becomes a tensor product mesh (only with a multiplicity elevantion at some of its boundary), and so we could very well start with $\mathcal{N}'_0$ instead of $\mathcal{N}'_{-1}$.

10 The case $d = 2$

For a mesh $\mathcal{N} = (\mathcal{M}, \mu, (p_1, p_2)$ in $\mathbb{R}^2$, the dimesion formulas from Theorem 2 are given as

$$\begin{aligned}\dim \mathbb{S}(\mathcal{N}) = {} & f_{\{1,2\}}(p_1+1)(p_2+1) \\ & - (p_1+1)\left(\sum_{\beta\in\mathcal{F}_{\{1\}}(\mathcal{M})}(p_2-\mu(\beta)+1)\right) \\ & - (p_2+1)\left(\sum_{\beta\in\mathcal{F}_{\{2\}}(\mathcal{M})}(p_1-\mu(\beta)+1)\right) \\ & + \sum_{\beta\in\mathcal{F}_0(\mathcal{M})}(p_1-\mu_1(\beta)+1)(p_2-\mu_2(\beta)+1) \\ & + \dim H_1(\mathfrak{S}(\mathcal{N})) - \dim H_0(\mathfrak{S}(\mathcal{N}))\end{aligned} \tag{156}$$

and, in case $\mathcal{N}$ is open,

$$\begin{aligned}\dim \mathbb{S}(\mathcal{N}) = f_{\{1,2\}}(p_1+1)(p_2+1) \\ -(p_1+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{1\}}(\mathcal{M})}(p_2-\mu(\beta)+1)\right) \\ -(p_2+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{2\}}(\mathcal{M})}(p_1-\mu(\beta)+1)\right) \\ +\sum_{\beta\in\mathcal{F}^0_0(\mathcal{M})}(p_1-\mu_1(\beta)+1)(p_2-\mu_2(\beta)+1) \\ +\dim H_1(\mathfrak{S}^0(\mathcal{N}))\end{aligned} \tag{157}$$

In the special case when $\mathcal{N}$ is open, and every inner 1-meshrectangle (horizontal line segment) has multiplicity m_1 and every inner 2-meshrectangle (vertical line segment) has multiplicity m_2, the dimension is given by Corollary 1 as

$$\begin{aligned}\dim \mathbb{S}(\mathcal{N}) = f_{\{1,2\}}(p_1+1)(p_2+1) \\ -f_{\{1\}}(p_1+1)(p_2-m_2+1)-f_{\{2\}}(p_1-m_1+1)(p_2+1) \\ +f_{\emptyset}(p_1-m_1+1)(p_2-m_2+1)+\dim H_1(\mathfrak{S}^0(\mathcal{N}))\end{aligned} \tag{158}$$

Definition 31. *Let $\mathcal{N}=(\mathcal{M},\mu,(p_1,p_2)$ be an open spline mesh in $\mathbb{R}^2$. Assume the line segment B is either a union of vertical line segments in $\mathcal{M}$ with uniform multiplicity $\le p_1$ or a union of horizontal line segments in $\mathcal{M}$ with uniform multiplicity $\le p_2$, or a line inside the domain $\Omega(\mathcal{M})$ not containing any line segment in $\mathcal{M}$. We consider the last case as a special case of the two first cases, by adding lines of multiplicity 0 to $\mathcal{N}$.*

Let $\gamma_1,\ldots,\gamma_n$ be the points in $\mathcal{F}_0(\mathcal{M})$ contained in B. We define

$$\alpha(\mathcal{N},B)=\begin{cases}p_2+1-\sum_{i=1}^n\mu_1(\gamma_i) & \text{if } B \text{ is vertical}\\ p_1+1-\sum_{i=1}^n\mu_2(\gamma_i) & \text{if } B \text{ is horizontal}\end{cases} \tag{159}$$

The line segment B is an **interior** *line segment if both of its endpoints lie in the interior of $\Omega(\mathcal{M})$. It is an* **extension** *if, for at least on of its end points γ, we have $\mu_1(\gamma)>\mu(B)$ if B is horizontal, or $\mu_2(\gamma)>\mu(B)$ if B is vertical. This means that there is a continuation of B after one of its endpoints with higher multiplicity than B.*

The term $\dim H_0(\mathfrak{S}(\mathcal{N}))$ in (156) can be calculated combinatorically by Lemma 17, and is zero unless the mesh has a very poor structure.

An uppoer bound of term $\dim H_1(\mathfrak{S}(\mathcal{N}))$ in (156) and (157) is given by

Lemma 20. *Let $\mathcal{N}_0 = (\mathcal{M}, \mu, (p_1, p_2))$ be a tensor product spline mesh in $\mathbb{R}^2$ and let $B_1, \ldots, B_n$ be a sequence of boxmeshes in $\mathcal{N}$. Define $\mathcal{N}_i = \mathcal{N}_{i-1} + B_i$ for $i = 1, \ldots, n$ and assume B_i has proper uniform multiplicity in $\mathcal{N}_{i-1}$ for every i. Let $\mathcal{N} = \mathcal{N}_n$. Also define $\phi_k = p_k + 1 - m_k$ for $k = 1, 2$ where m_k is the length of the knot vector defining the spline space in parameter direction k. Then*

$$\dim H_1(\mathfrak{S}(\mathcal{N})) \leq \max(0, -\phi_1\phi_2) + \sum_i \max(0, \alpha(\mathcal{N}_{i-1}, B_i) \tag{160}$$

where the sum is taken only over the i such that B_i is not an extension of $\mathcal{N}_{i-1}$. If $\mathcal{N}_0$ is open, then so is $\mathcal{N}$, and we have

$$\dim H_1(\mathfrak{S}(\mathcal{N})) = \dim H_1(\mathfrak{S}^0(\mathcal{N})) \leq \sum_i \max(0, \alpha(\mathcal{N}_{i-1}, B_i) \tag{161}$$

where the sum is taken only over the i such that B_i is interior and not an extension of $\mathcal{N}_{i-1}$.

Proof. We use Theorem 6, where $\dim H_1(\mathfrak{S}(\mathcal{N}_0)) = \max(0, -\phi_1\phi_2)$ by Corollary 2. The mesh $S(\mathcal{N}_{i-1} \cap B_i$ is univariate, and so its homology terms is given by Lemma 15. If B is an extension, the multiplicity in one of the end points of $\mathcal{N}_{i-1} \cap B_i$ is polynomial degree $+1$, then $H_0(S(\mathcal{N}_{i-1} \cap B_i) = 0$, so we can ommit this term.

In case $\mathcal{N}_0$ is open, we use Corollary 7 and the fact that if B is not interior, then again the multiplicity in one of the end points of $\mathcal{N}_{i-1} \cap B_i$ is polynomial degree $+1$. □

11 The case $d = 3$

For a mesh $\mathcal{N} = (\mathcal{M}, \mu, (p_1, p_2, p_3)$ in $\mathbb{R}^3$, the dimesion formulas from Theorem 2 are given as

$$\begin{aligned}
\dim \mathbb{S}(\mathcal{N}) = {} & f_{\{1,2,3\}}(p_1+1)(p_2+1)(p_3+1) \\
& - (p_1+1)(p_2+1)\left(\sum_{\beta \in \mathcal{F}_{\{1,2\}}(\mathcal{M})} (p_3 - \mu(\beta) + 1)\right) \\
& - (p_1+1)(p_3+1)\left(\sum_{\beta \in \mathcal{F}_{\{1,3\}}(\mathcal{M})} (p_2 - \mu(\beta) + 1)\right) \\
& - (p_2+1)(p_3+1)\left(\sum_{\beta \in \mathcal{F}_{\{2,3\}}(\mathcal{M})} (p_1 - \mu(\beta) + 1)\right) \\
& + (p_1+1)\left(\sum_{\beta \in \mathcal{F}_{\{1\}}(\mathcal{M})} (p_2 - \mu_2(\beta) + 1)(p_3 - \mu_3(\beta) + 1)\right) \\
& + (p_2+1)\left(\sum_{\beta \in \mathcal{F}_{\{2\}}(\mathcal{M})} (p_1 - \mu_1(\beta) + 1)(p_3 - \mu_3(\beta) + 1)\right) \\
& + (p_3+1)\left(\sum_{\beta \in \mathcal{F}_{\{3\}}(\mathcal{M})} (p_1 - \mu_1(\beta) + 1)(p_2 - \mu_2(\beta) + 1)\right) \\
& - \sum_{\beta \in \mathcal{F}_0(\mathcal{M})} (p_1 - \mu_1(\beta) + 1)(p_2 - \mu_2(\beta) + 1)(p_3 - \mu_3(\beta) + 1) \\
& + \dim H_2(\mathfrak{S}(\mathcal{N})) - \dim H_1(\mathfrak{S}(\mathcal{N})) + \dim H_0(\mathfrak{S}(\mathcal{N}))
\end{aligned} \tag{162}$$

and, in case $\mathcal{N}$ is open,

$$\begin{aligned}
\dim \mathbb{S}(\mathcal{N}) = f_{\{1,2,3\}}&(p_1+1)(p_2+1)(p_3+1) \\
&- (p_1+1)(p_2+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{1,2\}}(\mathcal{M})}(p_3-\mu(\beta)+1)\right) \\
&- (p_1+1)(p_3+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{1,3\}}(\mathcal{M})}(p_2-\mu(\beta)+1)\right) \\
&- (p_2+1)(p_3+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{2,3\}}(\mathcal{M})}(p_1-\mu(\beta)+1)\right) \\
&+ (p_1+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{1\}}(\mathcal{M})}(p_2-\mu_2(\beta)+1)(p_3-\mu_3(\beta)+1)\right) \\
&+ (p_2+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{2\}}(\mathcal{M})}(p_1-\mu_1(\beta)+1)(p_3-\mu_3(\beta)+1)\right) \\
&+ (p_3+1)\left(\sum_{\beta\in\mathcal{F}^0_{\{3\}}(\mathcal{M})}(p_1-\mu_1(\beta)+1)(p_2-\mu_2(\beta)+1)\right) \\
&- \sum_{\beta\in\mathcal{F}^0_{0}(\mathcal{M})}(p_1-\mu_1(\beta)+1)(p_2-\mu_2(\beta)+1)(p_3-\mu_3(\beta)+1) \\
&+ \dim H_2(\mathfrak{S}^0(\mathcal{N})) - \dim H_1(\mathfrak{S}^0(\mathcal{N}))
\end{aligned} \tag{163}$$

In the special case when $\mathcal{N}$ is open, and every inner k-meshrectangle has multiplicity m_k for $k = 1, 2, 3$, the dimension is given by Corollary 1 as

$$\begin{aligned}
\dim \mathbb{S}(\mathcal{N}) = & f_{\{1,2,3\}}(p_1+1)(p_2+1)(p_3+1) \\
& - f_{\{1,2\}}(p_1+1)(p_2+1)(p_3-m_3+1) \\
& - f_{\{1,3\}}(p_1+1)(p_2-m_2+1)(p_3+1) \\
& - f_{\{2,3\}}(p_1-m_1+1)(p_2+1)(p_3+1) \\
& + f_{\{1\}}(p_1+1)(p_2-m_2+1)(p_3-m_3+1) \\
& + f_{\{2\}}(p_1-m_1+1)(p_2+1)(p_3-m_3+1) \\
& + f_{\{3\}}(p_1-m_1+1)(p_2-m_2+1)(p_3+1) \\
& - f_{\emptyset}(p_1-m_1+1)(p_2-m_2+1)(p_3-m_3+1) \\
& + \dim H_2(\mathfrak{S}^0(\mathcal{N})) - \dim H_1(\mathfrak{S}^0(\mathcal{N}))
\end{aligned} \tag{164}$$

Again, the term $\dim H_0(\mathfrak{S}(\mathcal{N}))$ in (156) can be calculated combinatorically by Lemma 17, and is zero unless the mesh has a very poor structure.

The terms $\dim H_q(\mathfrak{S}(\mathcal{N}))$ and $\dim H_q(\mathfrak{S}^0(\mathcal{N}))$ for $q = 1, 2$ in the hierarchical case can be limited by homologies from 2-dimensional meshes using Theorem 6 and Corollary 7. Just as for the case $d = 2$, we can simplify for $q = 1$ when we have extensions in both directions, this time meaning that we omit terms where each of the two parameter diretions in the 2-dimensional k-meshblock B_i have at least one edge E such that either $\mu_k(\beta) > \mu(B)$ for every line segment $\beta \in \mathcal{F}_1(\mathcal{M})$ on E, or $E \subseteq \partial\Omega(\mathcal{M})$ and $\mathcal{N}_0$ is open.

It is important that the B_i slot into the $\mathcal{N}_{i-1}$. In picture number two from the left in Figure 5, calculation will show that all homologies of $\mathcal{N}_0$, $\mathcal{N}_0 \cap C$ and $(\mathcal{N}_0 \cap C) \cap B$ are zero, yet still the final mesh $(\mathcal{N}_0 + C) + B$ has a non-trivial homology group.

References

[1] T. Dokken, T. Lyche, K. F. Pettersen, Polynomial Splines over Locally Refined Box-Partitions, accepted for publication in Computer Aided Geometric Design, 2013

[2] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002

[3] B. Mourrain, On the dimension of spline spaces on planar T-subdivisions, Preprint, 2010. http://hal.inria.fr/inria-00533187/en