Approximate Implicitization

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Parametric and Implicit Representations

We will address rational *parametric* surfaces

$$S_P = \left\{ \mathbf{p}(s_1, s_2) \in \mathbb{R}^3 : (s_1, s_2) \in \Omega \subseteq \mathbb{R}^2 \right\},$$

where $\mathbf{p}(s_1, s_2) = (p_1(s_1, s_2), p_2(s_1, s_2), p_3(s_1, s_2))$, and $p_1(s_1, s_2)$, $p_2(s_1, s_2)$, and $p_3(s_1, s_2)$ are bivariate polynomials or rational functions with the same denominator of degree $\mathbf{n}$.

We will approximate these with *implicit* surfaces defined as the zero set of a nontrivial trivariate polynomial q of degree $m > 0$:

$$S_I = \left\{ (x, y, z) \in \mathbb{R}^3 : q(x, y, z) = 0 \right\}.$$

Exact and Approximate Implicitization

A nontrivial polynomial q gives an exact implicitization of $\mathbf{p}(\mathbf{s})$ if

$$q(\mathbf{p}(\mathbf{s})) = 0, \text{ for all } \mathbf{s} \in \Omega.$$

A nontrivial polynomial q gives an approximate implicitization of $\mathbf{p}(\mathbf{s})$ within the tolerance ε if there exists $\mathbf{g}(\mathbf{s})$ such that

$$q(\mathbf{p}(\mathbf{s}) + \mathbf{g}(\mathbf{s})) = 0, \text{ for all } \mathbf{s} \in \Omega,$$

and

$$\max_{\mathbf{s} \in \Omega} \|\mathbf{g}(\mathbf{s})\| \leq \varepsilon.$$

Applications of the Implicitization

Applications:

- Intersection algorithms - detecting self-intersections,
- Ray tracing,
- Classification - is a given point above, below or on the surface.

Approaches Approximate Implicitization:

- Dokken 1997 (Strong form),
- Sederberg et al. 1999 (Monoids),
- Wurm and Jüttler 2003 (Point based curves),
- Dokken and Thomassen 2006 (Weak form),
- Barrowclough and Dokken 2010 (Triangular Bézier, General Point Based).

Barycentric coordinates

Barycentric coordinates allow us to express any point $\mathbf{x} \in \mathbb{R}^l$ as,

$$\mathbf{x} = \sum_{i=1}^{l+1} \beta_i \mathbf{a}_i,$$

where $\mathbf{a}_i \in \mathbb{R}^l$ are points defining the vertices of a non-degenerate simplex in $\mathbb{R}^l$ under the condition that $\sum_{i=1}^{l+1} \beta_i = 1$,

- In $\mathbb{R}^2$ the simplex is a triangle
- In $\mathbb{R}^3$ the simplex is a tetrahedron

In the reminder of the presentation we assume that the Bézier curves and surfaces are inside the simplex so all vertices expressed in barycentric coordinates are positive.

Multi index and vector notation

- We will use a vector and multi index notation for describing the rational parametric objects
- This allows us to describe the approach in a generic way

$$\mathbf{p}(\mathbf{s}) = \sum_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} \mathbf{c}_{\mathbf{i}} B_{\mathbf{i},\mathbf{n}}(\mathbf{s}),\ \mathbf{s} \in \Omega,$$

- where the basis functions $B_{\mathbf{i},\mathbf{n}}(\mathbf{s})$, $\mathbf{i} \in \mathcal{I}_{\mathbf{n}}$, satisfy the partition of unity property

$$\begin{aligned} \sum_{\mathbf{i} \in \mathcal{I}_{\mathbf{n}}} B_{\mathbf{i},\mathbf{n}}(\mathbf{s}) &= 1,\ \mathbf{s} \in \Omega,\ \text{with} \\ B_{\mathbf{i},\mathbf{n}}(\mathbf{s}) &\geq 0,\ \mathbf{s} \in \Omega,\ \mathbf{i} \in \mathcal{I}_{\mathbf{n}}. \end{aligned}$$

- The coefficients are represented in projective space $\mathbf{c}_{\mathbf{i}} \in \mathbb{P}$, $\mathbf{i} \in \mathcal{I}_{\mathbf{n}}$, to also support rational parametrization.

Barycentric coordinates and Bernstein bases over Simplices

For bivariate barycentric coordinates $\mathbf{s} = (s_1, s_2, s_3)$, the triangular Bernstein basis polynomials of degree n are:

$$B_{\mathbf{i}}^{n}(\mathbf{s}) = \binom{n}{i_1, i_2, i_3} s_1^{i_1} s_2^{i_2} s_3^{i_3}, \quad |\mathbf{i}| = i_1 + i_2 + i_3 = n.$$

For trivariate barycentric coordinates $\mathbf{u} = (u_1, u_2, u_3, u_4)$; the tetrahedral Bernstein basis polynomials of degree m are:

$$B_{\mathbf{i}}^{m}(\mathbf{u}) = \binom{m}{i_1, i_2, i_3, i_4} u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4}, \quad |\mathbf{i}| = i_1 + i_2 + i_3 + i_4 = m.$$

Multinomial coefficients:

$$\binom{n}{i_1, i_2, i_3} = \frac{n!}{i_1! i_2! i_3!}.$$

Example: Bézier Curves and Surface

- For 2D Bézier curves $l = 2$, $\mathbf{n} = n$, $\mathbf{i} = i$, $\mathbf{c_i} = (c_{i,x}, c_{i,y}, c_{i,h})$ and $\mathcal{I} = \{0, \ldots, n\}$, giving

$$\mathbf{p}(s) = \sum_{i=0}^{n} \mathbf{c}_i \binom{n}{i} (1-s)^{n-i} s^i .$$

- For tensor product Bézier surfaces $l = 3$, $\mathbf{n} = (n_1, n_2)$, $\mathbf{i} = (i_1, i_2)$, $\mathbf{c_i} = (c_{\mathbf{i},x}, c_{\mathbf{i},y}, c_{\mathbf{i},z}, c_{\mathbf{i},h})$ and $\mathcal{I} = \{(i_1, i_2) \mid 0 \leq i_1 \leq n_1 \wedge 0 \leq i_2 \leq n_2\}$, giving,

$$\mathbf{p}(\mathbf{s}) = \sum_{i_1=0}^{n_1} \sum_{i_2=0}^{n_2} \mathbf{c}_{i_1,i_2} \binom{n_1}{i_1} \binom{n_2}{i_2} (1-s_1)^{n_1-i_1} s_1^{i_1} (1-s_2)^{n_2-i_2} s_2^{i_2} .$$

Example: Triangular Bézier Surface

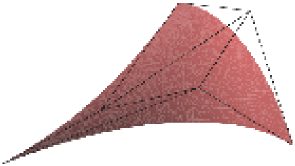
For triangular Bézier surfaces $l = 3$, $\mathbf{n} = n$, $\mathbf{i} = (i_1, i_2, i_3)$, $\mathbf{c_i} = (c_{\mathbf{i},x}, c_{\mathbf{i},y}, c_{\mathbf{i},z}, c_{\mathbf{i},h})$ and

$$\mathcal{I}_n = \{(i_1, i_2, i_3) \mid 0 \leq i_1, i_2, i_3 \leq n \wedge i_1 + i_2 + i_3 = n\},$$

giving,

$$\mathbf{p}(\mathbf{s}) = \sum_{i_1+i_2+i_3=n} \mathbf{c}_{i_1,i_2,i_3} \binom{n}{i_1\ i_2\ i_3} s_1^{i_1} s_2^{i_2} s_3^{i_3},$$

with (s_1, s_2, s_3) *barycentric* coordinates in $\Omega \subset \mathbb{R}^2$.



Implicit Surfaces and Algebraic Distance

The intention is to find a polynomial q describing an implicit surface that approximates $\mathbf{p}(\mathbf{s})$ in the tetrahedral Bernstein basis of degree m

$$q(\mathbf{u}) = \sum_{|\mathbf{i}|=m} b_{\mathbf{i}} B^m_{\mathbf{i}}(\mathbf{u}).$$

The task is to find the unknown values $b_{\mathbf{i}}$ for $|\mathbf{i}| = m$ that satisfy the approximation criteria.

The algebraic distance between an implicitly defined surface and a point $\mathbf{u}_0 \in \mathbb{P}^3$ is the value $q(\mathbf{u}_o)$.

Implicitization and exact degrees

- A rational parametric 2D curve of degree n has an implicit degree of at most $m = n$.
- A parametric surfaces of bi-degree (n_1, n_2) has an implicit degree of at most $m = 2n_1 n_2$.
- A parameter surface of total degree n has an implicit degree of at most $m = n^2$.

Approximate implicitization allows algebraic curve and surface approximations with lower degrees than the exact degree m while using floating point arithmetic.

Approximate Implicitization

We attempt to minimize the algebraic distance, given $\mathbf{p}(\mathbf{s})$ and a chosen degree m of q:

- *Original Approach*: Minimize the algebraic distance point wise:

$$\min_{\|\mathbf{b}\|=1} \max_{\mathbf{s}\in\Omega} \left| q(\mathbf{p}(\mathbf{s})) \right| ,$$

- *Weak Approach*: Minimize the integral of the squared algebraic distance:

$$\min_{\|\mathbf{b}\|=1} \int_{\Omega} \left(q(\mathbf{p}(\mathbf{s})) \right)^2 d\mathbf{s},$$

- *Point based approach*: Minimize the squared algebraic distance for a set of points $\mathbf{p}(\mathbf{s}_k)$, $k = 1, \ldots, N$

$$\min_{\|\mathbf{b}\|=1} \sum_{k=1}^{N} \left(q(\mathbf{p}(\mathbf{s}_k)) \right)^2 .$$

Original Approach

Define a matrix $\mathbf{D}$ by the values $(d_{\mathbf{i},\mathbf{j}})_{|\mathbf{i}|=m,\mathbf{j}\in\mathcal{J}_{m,\mathbf{n}}}$, a vector $\mathbf{B}^{m\mathbf{n}}(\mathbf{s}) = (\mathbf{B}^{m\mathbf{n}}(\mathbf{s}))_{\mathbf{j}\in\mathcal{J}_{m,\mathbf{n}}}$ and a vector $\mathbf{b} = (b_{\mathbf{i}})_{|\mathbf{i}|=m}$.

$$
\begin{aligned}
q(\mathbf{p}(\mathbf{s})) &= \sum_{|\mathbf{i}|=m} b_{\mathbf{i}} B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s})) \\
&= \sum_{|\mathbf{i}|=m} b_{\mathbf{i}} \left(\sum_{\mathbf{j}\in\mathcal{J}_{m,\mathbf{n}}} d_{\mathbf{i},\mathbf{j}} B_{\mathbf{j}}^{m\mathbf{n}}(\mathbf{s}) \right) = \sum_{\mathbf{j}\in\mathcal{J}_{m,\mathbf{n}}} B_{\mathbf{j}}^{m\mathbf{n}}(\mathbf{s}) \left(\sum_{|\mathbf{i}|=m} d_{\mathbf{i},\mathbf{j}} b_{\mathbf{i}} \right) \\
&= \mathbf{B}^{m\mathbf{n}}(\mathbf{s})^{T} \mathbf{D}\mathbf{b}.
\end{aligned}
$$

Let $\sigma_{\min}$ be the smallest singular value of $\mathbf{D}$.

$$
\min_{\|\mathbf{b}\|=1} \max_{\mathbf{s}\in\Omega} |q(\mathbf{p}(\mathbf{s}))| \leq \max_{\mathbf{s}\in\Omega} \|\mathbf{B}^{m\mathbf{n}}(\mathbf{s})\|_2 \min_{\|\mathbf{b}\|=1} \|\mathbf{D}\mathbf{b}\|_2 \leq \sigma_{\min}.
$$

Original Approach

- For 2D curves $\mathcal{J}_{m,\mathbf{n}} = \mathcal{J}_{m,n} = \{1, \ldots, mn\}$
- For triangular Bézier surfaces $\mathcal{J}_{m,\mathbf{n}} = \mathcal{J}_{m,n} = \{\mathbf{j} : |\mathbf{j}| = mn\}$,
- For tensor Bézier surfaces

$$\mathcal{J}_{m,\mathbf{n}} = \mathcal{J}_{m,(n_1,n_2)} = \{(j_1, j_2) : 1 \leq j_1 \leq mn_1 \wedge 1 \leq j_2 \leq mn_2\}.$$

To summarize the approach:

- To produce $\mathbf{D}$ multiply out the coordinate functions of $\mathbf{p}(\mathbf{s})$ according to

$$B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s})) = \left(\sum_{\mathbf{j} \in \mathcal{J}_{m,\mathbf{n}}} d_{\mathbf{i},\mathbf{j}} B_{\mathbf{j}}^{m\mathbf{n}}(\mathbf{s}) \right).$$

- Perform SVD on $\mathbf{D} = (d_{\mathbf{i},\mathbf{j}})_{|\mathbf{i}|=m, \mathbf{j} \in \mathcal{J}_{m,\mathbf{n}}}$.
- Choose the coefficients $\mathbf{b}_{\min} = (b_{\mathbf{i}})_{|\mathbf{i}|=m}$ corresponding to the smallest singular value $\sigma_{\min}$ in the SVD as the solution to the approximation problem.

Weak Approach

$$\begin{aligned}
\int_{\Omega} (q(\mathbf{p}(\mathbf{s})))^2 \, d\mathbf{s} &= \mathbf{b}^T \mathbf{D}^T \left(\int_{\Omega} \mathbf{B}^{m\mathbf{n}}(\mathbf{s}) \mathbf{B}^{m\mathbf{n}}(\mathbf{s})^T \, d\mathbf{s} \right) \mathbf{D}\mathbf{b}. \\
&= \mathbf{b}^T \mathbf{D}^T \mathbf{A} \mathbf{D} \mathbf{b} \\
&= \mathbf{b}^T \mathbf{D}^T \mathbf{U}^T \Sigma \Sigma \mathbf{U} \mathbf{D} \mathbf{b} \\
&= \| \Sigma \mathbf{U} \mathbf{D} \mathbf{b} \|_2^2 .
\end{aligned}$$

with $\mathbf{A} = \mathbf{U}^T \Sigma \Sigma \mathbf{U}$ the singular value decomposition of $\mathbf{A}$. For the triangular Bézier surface $\mathbf{A}$ looks like.

$$\begin{aligned}
a_{i,j} &= \int_{\Omega} B_{\mathbf{i}}^{mn}(\mathbf{s}) B_{\mathbf{j}}^{mn}(\mathbf{s}) d\mathbf{s} = \frac{\binom{mn}{\mathbf{i}}\binom{mn}{\mathbf{j}}}{\binom{2mn}{\mathbf{i}+\mathbf{j}}} \int_{\Omega} B_{\mathbf{i}+\mathbf{j}}^{2mn}(\mathbf{s}) d\mathbf{s} \\
&= \frac{\binom{mn}{\mathbf{i}}\binom{mn}{\mathbf{j}}}{\binom{2mn}{\mathbf{i}+\mathbf{j}}} \frac{1}{(2mn+1)(2mn+2)}.
\end{aligned}$$

Numerical Weak Approach

The integral in weak approximate implicitization can also be evaluated numerically. Using the factorization

$$q(\mathbf{p}(\mathbf{s})) = \sum_{|\mathbf{i}|=m} b_{\mathbf{i}} B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s}))$$

$$\int_{\Omega} (q(\mathbf{p}(\mathbf{s})))^2 \, d\mathbf{s} = \int_{\Omega} \left(\sum_{|\mathbf{i}|=m} b_{\mathbf{i}} B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s})) \right)^2 d\mathbf{s} = \mathbf{b}^T \mathbf{M} \mathbf{b}$$

$$m_{\mathbf{i},,\mathbf{j}} = \int_{\Omega} B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s})) B_{\mathbf{j}}^{m}(\mathbf{p}(\mathbf{s})) d\mathbf{s} = \frac{\binom{m}{\mathbf{i}}\binom{m}{\mathbf{j}}}{\binom{2m}{\mathbf{i}+\mathbf{j}}} \int_{\Omega} B_{\mathbf{i}+\mathbf{j}}^{2m}(\mathbf{p}(\mathbf{s})) d\mathbf{s}.$$

Exploiting symmetries within this algorithm can significantly reduce the computation time.

Point Based Approach

Let $v(\mathbf{i})$ be a lexicographical ordering such that $B_{\mathbf{i}}^m(\mathbf{s}) = B_{v(\mathbf{i})}^m(\mathbf{s})$, $b_{\mathbf{i}} = b_{v(\mathbf{i})}$ and let $L = \binom{m+3}{3}$ be the number of basis functions

$$\begin{aligned}
\sum_{k=1}^{N} \left(q(\mathbf{p}(\mathbf{s}_k))\right)^2 &= \sum_{k=1}^{N} \left(\sum_{i=1}^{L} b_i B_i^m(\mathbf{p}(\mathbf{s}_k)) \right)^2 \\
&= \left\| \begin{pmatrix} B_1^m(\mathbf{p}(\mathbf{s}_1)) & \dots & B_L^m(\mathbf{p}(\mathbf{s}_1)) \\ \vdots & & \vdots \\ B_1^m(\mathbf{p}(\mathbf{s}_k)) & \dots & B_L^m(\mathbf{p}(\mathbf{s}_k)) \\ & & \vdots \\ B_1^m(\mathbf{p}(\mathbf{s}_N)) & \dots & B_L^m(\mathbf{p}(\mathbf{s}_N)) \end{pmatrix} \mathbf{b} \right\|_2^2 \\
&= \|\mathbf{C}\mathbf{b}\|_2^2 = \mathbf{b}^T \mathbf{C}^T \mathbf{C}\mathbf{b}.
\end{aligned}$$

Point Based Approach

Now looking at column $\mathbf{c}_i$ the i^{th} column of $\mathbf{C}$

$$(\mathbf{c}_i)^T = \left(\; B_i^m(\mathbf{p}(\mathbf{s}_1)) \quad \ldots \quad B_i^m(\mathbf{p}(\mathbf{s}_k)) \quad \ldots \quad B_i^m(\mathbf{p}(\mathbf{s}_N)) \; \right).$$

The polynomial $B_i^m(\mathbf{p}(\mathbf{s}))$ is a polynomial $B_i^{m\mathbf{n}}(\mathbf{s})$ of degree $m\mathbf{n}$ in the variables $\mathbf{s}$. The number of basis functions K in the polynomial space used for describing $B_i^{m\mathbf{n}}(\mathbf{s})$ depends on $\mathbf{p}(\mathbf{s})$ being a curve, a tensor product Bézier surface or a triangular Bézier surface:

- In the curve case $K = mn + 1$.
- In the tensor product Bézier surface case $K = (mn_1 + 1)\,(mn_2 + 1)$.
- In the triangular Bézier surface case $K = \binom{mn+2}{2}$.

Point Based Approach

Now choosing the number of interpolation points to $N = K$ we can pose interpolation problems using the basis functions $B_{\mathbf{j}}^{m\mathbf{n}}(\mathbf{s})$ from the original approach to reproduce $B_{\mathbf{i}}^{m}(\mathbf{p}(\mathbf{s}))$ and its coefficients $\mathbf{d_i}$

$$\begin{aligned} \mathbf{G}\mathbf{d_i} &= \mathbf{c_i}, \text{ with} \\ \mathbf{G} &= \left(B_{\mathbf{j}}^{m\mathbf{n}}(\mathbf{s}_k)\right)_{\mathbf{j}\in\mathcal{J}_{m,\mathbf{n}},k=1,\dots,K} . \end{aligned}$$

Provided $\mathbf{G}$ is nonsingular the rows $\mathbf{d_i}$ of the matrix $\mathbf{D} = (\mathbf{d_i})$ of the original approach can be expressed

$$\mathbf{d_i} = \mathbf{G}^{-1}\mathbf{c_i}.$$

Using this we get

$$\begin{aligned} \sqrt{\sum_{k=1}^{N}\left(q(\mathbf{p}(\mathbf{s}_k))\right)^2} &= \|\mathbf{C}\mathbf{b}\|_2 \\ &= \left\|\mathbf{G}\mathbf{G}^{-1}\mathbf{C}\mathbf{b}\right\|_2 = \|\mathbf{G}\mathbf{D}\mathbf{b}\|_2 . \end{aligned}$$

Relations between Approaches

Let $\sigma_{\min}$ be the smallest singular value of $\mathbf{D}$.

- *Original Approach*:

$$\min_{\|\mathbf{b}\|=1} \max_{\mathbf{s}\in\Omega} |q(\mathbf{p}(\mathbf{s}))| = \min_{\|\mathbf{b}\|=1} \|\mathbf{D}\mathbf{b}\|_2 \leq \sigma_{\min}.$$

- *Weak Approach*: Let $\lambda_{\max}$ be the largest eigenvalue of Σ

$$\min_{\|\mathbf{b}\|=1} \sqrt{\int_{\Omega} (q(\mathbf{p}(\mathbf{s})))^2 \, d\mathbf{s}} = \min_{\|\mathbf{b}\|=1} \|\Sigma\mathbf{U}\mathbf{D}\mathbf{b}\|_2 \leq \lambda_{\max}\sigma_{\min}.$$

- *Point based approach*: Let $\mathbf{G}$ be nonsingular and $g_{\max}$ its largest eigenvalue

$$\min_{\|\mathbf{b}\|=1} \sqrt{\sum_{k=1}^{N} (q(\mathbf{p}(\mathbf{s}_k)))^2} = \min_{\|\mathbf{b}\|=1} \|\mathbf{G}\mathbf{D}\mathbf{b}\| \leq g_{\max}\sigma_{\min}.$$

Convergence

- Curves in $\mathbb{R}^2$ are approximated with convergence

$$O(h^{\frac{(m+1)(m+2)}{2}}).$$

m	1	2	3	4	5	6
rate	2	5	9	14	20	35

- Surfaces in $\mathbb{R}^3$ are approximated with convergence

$$O(h^{\lfloor \frac{1}{6}\sqrt{(9+12m^3+72m^2+132m}-\frac{1}{2}\rfloor}).$$

m	1	2	3	4	5	6
rate	2	3	5	7	10	12

Singular Bézier Triangle

$$
\begin{aligned}
&c_{200} = (0,0,0), \\
&c_{110} = (0,0,1), \quad c_{101} = (0,1,0), \\
&c_{020} = (0,0,0), \quad c_{011} = (1,0,0), \quad c_{002} = (0,0,0).
\end{aligned}
$$

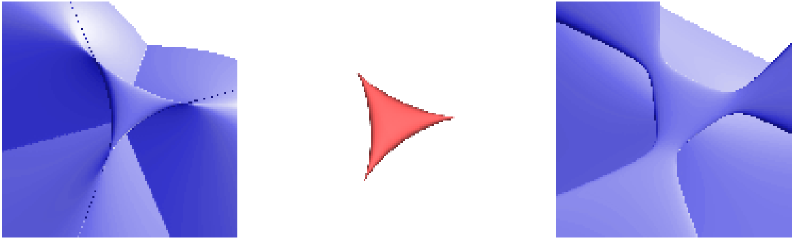
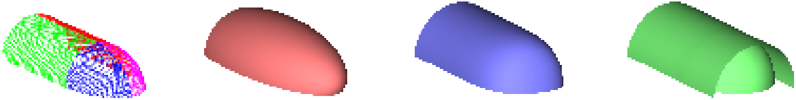


Figure: Exact (left) and approximate (right) implicitization of the parametric triangular Bézier surface (middle).

Several Patches Simultaneously



Parametric Quadratic Cubic Quartic

The original approach stacks the matrices:

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 \\ \vdots \\ \mathbf{D}_r \end{pmatrix}.$$

The weak and point based approaches sum the matrices:

$$\mathbf{M} = \sum_{i=1}^{r} \mathbf{M}_i,$$

$$\mathbf{C} = \sum_{i=1}^{r} \mathbf{C}_i.$$

Conclusion

- Approximate implicitization combines algebraic geometry, computer aided design and linear algebra to offer a family of methods for the approximation of parametric curves and surfaces by algebraic curves and surfaces.
- The methods have proven high convergence.
- The methods employ stable numerical methods.

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