Real time Ray-Casting of Algebraic Surfaces

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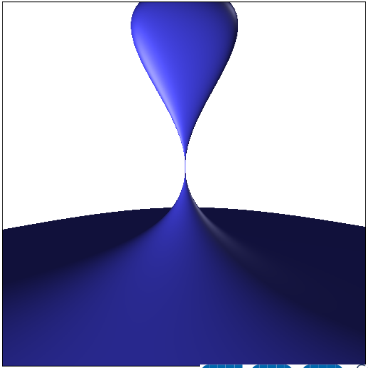
Raycasting

An algebraic surface is the zero set of a polynomial

$$f(x, y, z) = \sum_{0 \le i+j+k \le d} f_{ijk} x^i y^j z^k = 0.$$

Raycasting amounts to "shooting" rays inside a view frustum (VF) and determine if they intersect the surface.

- Can miss thin features
- Conceptually "easy"
- Embarrassingly parallel



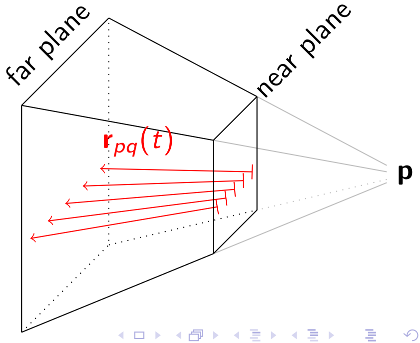
For a ray $\mathbf{r}_{pq}(t)$ raycasting amounts to finding the smallest $t \in [0, 1]$ s. t.

$$f((1-t)\mathbf{n}_{pq} + t\mathbf{f}_{pq}) = f(\mathbf{r}_{pq}(t)) = 0.$$

We would like to work on a univariate polynomial in Bernstein form,

$$f(\mathbf{r}_{pq}(t)) = \sum_{k=0}^{d} c_{pqk} B_k^d(t) = 0.$$

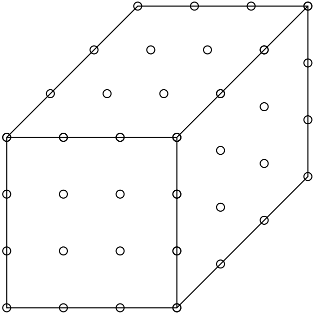
- Assume a screen resolution of $(m+1) \times (n+1)$ pixels.
- Pixel (p, q) corresponds to a ray through $\mathbf{p}$ and the pixel with coordinates $(p/m, q/n)$.



Overview

Our algorithm thus consist of the following steps:

1. Compute ray coefficients $C = (c_{pqk})$



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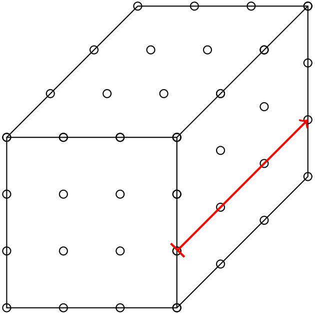
Our algorithm thus consist of the following steps:

1. Compute ray coefficients $C = (c_{pqk})$
2. For each pixel (p, q)
   2.1 Seek the smallest root $t \in [0, 1]$ of $f(\mathbf{r}_{pq}(t))$.
   2.2 Compute a color for pixel (p, q).



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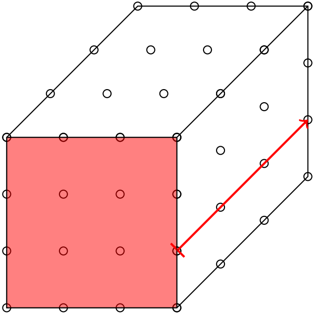
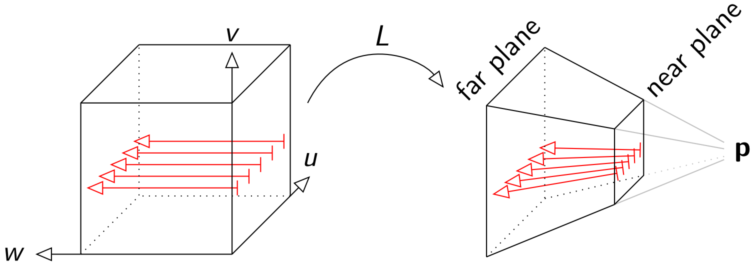
   2.1 Seek the smallest root $t \in [0, 1]$ of $f(\mathbf{r}_{pq}(t))$.
   2.2 Compute a color for pixel (p, q).
3. Optionally, perform postprocessing
   3.1 Detect singularities
   3.2 Antialias



The View Frustum Form

Idea: *Parameterize the view frustum over the unit cube, s. t.* $(u, v, 0)$ *and* $(u, v, 1)$ *maps to points on the near and far plane.*



A ray in the view frustum is given by: $\mathbf{r}_{pq}(w) = L(p/m, q/n, w)$.
We define the View Frustum Form to be:

$$g = f \circ L : [0, 1]^3 \rightarrow \mathbb{R}.$$

Using the composition $g = f \circ L$,

$$f(L(\frac{p}{m}, \frac{q}{n}, w)) = g(\frac{p}{m}, \frac{q}{n}, w) = \sum_{i,j,k=0}^{d,d,d} g_{ijk} B_i^d(\frac{p}{m}) B_j^d(\frac{q}{n}) B_k^d(w)$$

$$= \sum_{k=0}^{d} \underbrace{\left(\sum_{i,j=0}^{d,d} g_{ijk} B_i^d(\frac{p}{m}) B_j^d(\frac{q}{n}) \right)}_{c_{pqk}} B_k^d(w).$$

Yielding univariate ray equations of degree d,

$$f(\mathbf{r}_{pq}(t)) = \sum_{k=0}^{d} c_{pqk} B_k^d(t).$$

Computing VFF Coefficients

The VFF coefficients $G = (g_{ijk})$ can be found in a number of ways:

- Blossoming [DeRose et.al. 1993].
- Recursion [Sederberg/Zundel 1989].
- Interpolation (our preferred approach).
  - Choose $(d+1)^3$ distinct interpolation points (u_p, v_q, w_r) on a grid.
  - Solve

$$\sum_{i,j,k=0}^{d,d,d} g_{ijk} \underbrace{B_i^d(u_p)}_{\Omega_p} \overbrace{B_j^d(u_q)}^{\Omega_q} \underbrace{B_j^d(u_r)}_{\Omega_r} = f(L(u_p, v_q, w_r))$$

  - Needs inverse of Bernstein collocation matrices $\Omega_p = (B_i^d(u_p))$.
  - Use Chebyshev interpolation points for stability.
  - Not dependent on the representation of f.

Benefits of the View Frustum Form

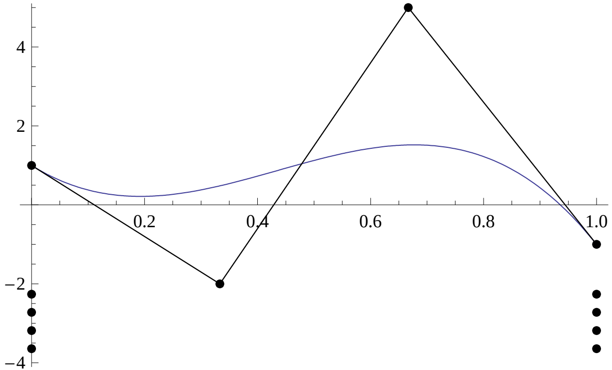
- For fixed m, n, d – precompute basis functions
  - $C_k = MG_kN^T, \quad M = (B_i^d(p/m)), N = (B_j^d(q/n))$.
  - Can pre-evaluate inverse of collocation matrix Ω^{-1}.
- Reduce algorithmic complexity
  - Evaluation of c_{pqk} requires $(d+1)^2$ muls/adds.
  - Evaluation of f requires $(d+1)(d+2)(d+3)/6$ muls/adds.
- Univariate ray equations for root finding.

$$f(\mathbf{r}_{pq}(t)) = \sum_{k=0}^{d} c_{pqk} B_k^d(t)$$

B-Spline Based Root Finding

The univariate ray equations can be expressed as B-Splines.

Idea: *Insert knot(s) at the first intersection of the control polygon. Repeat.*



Second order convergence for simple roots [Mørken and Reimers, 2006].

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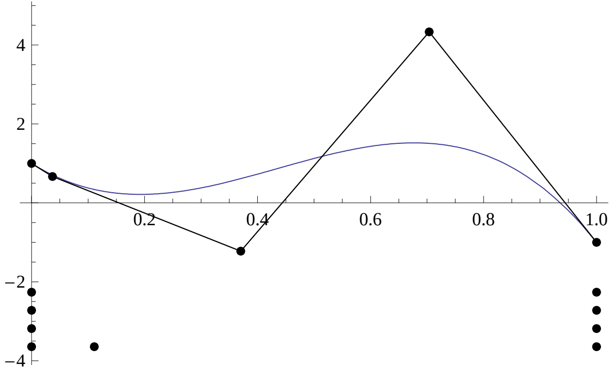

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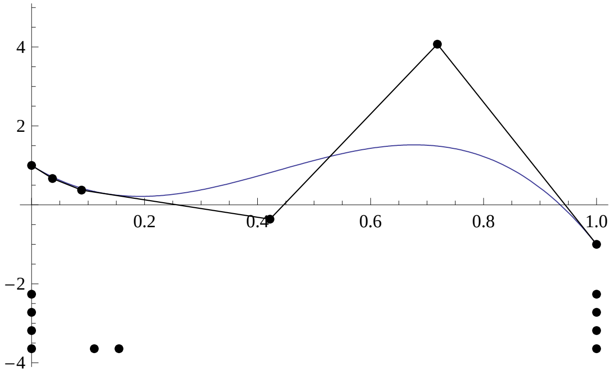

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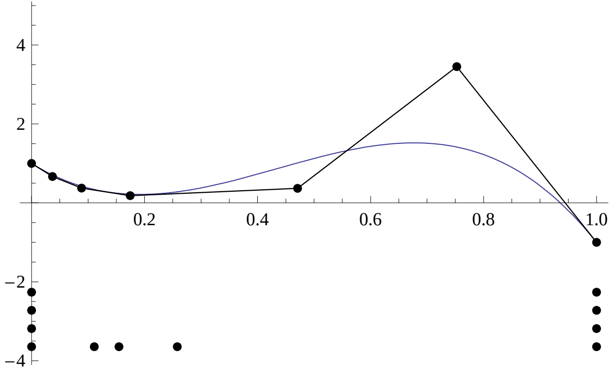

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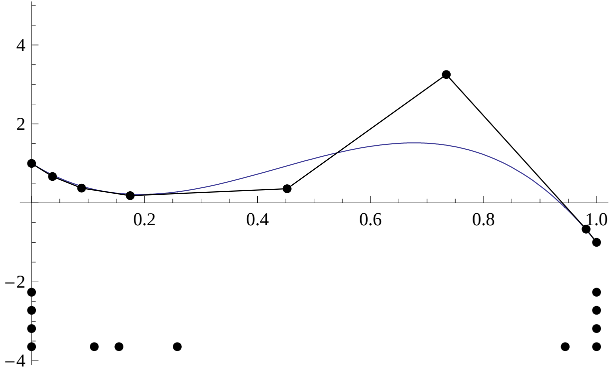

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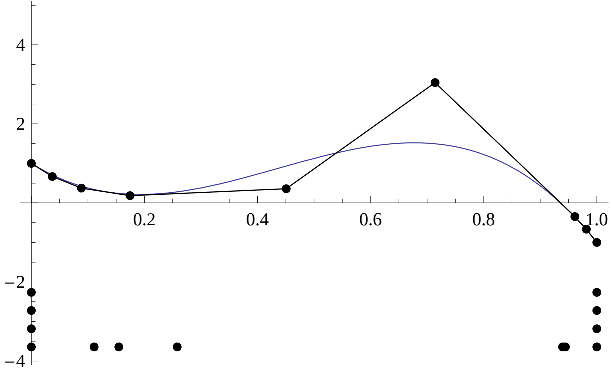

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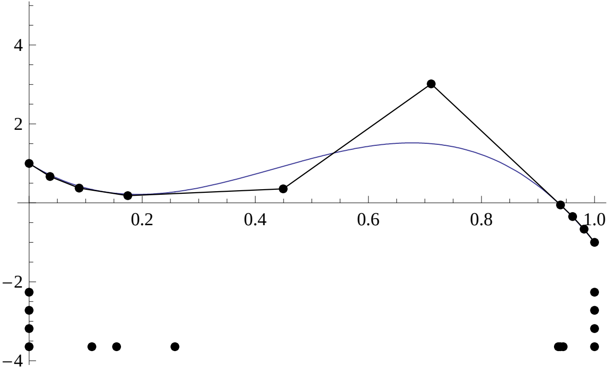

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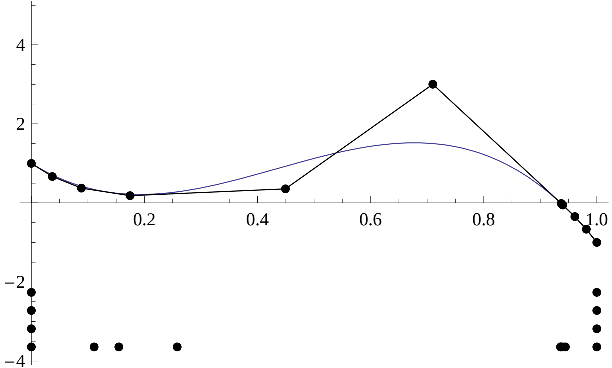

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Properites of rootfinder

- Stable computations (convex combinations)
- Similar to Newtons, but unconditionally convergent to smallest zero (no guessing)
- Quadratic convergence rate to simple zeros
- Recent extension [Mørken/Reimers] converge quadratically to multiple zeros
- Similar method [Reimers] for computing e.g. $\max f(x)$ or $\min |f(x)|$

Root finding variations

The knot insertion framework is very flexible, and allow for variations:

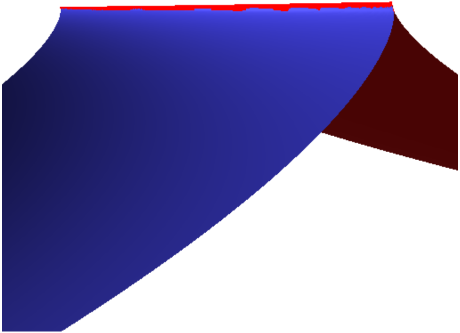
- Can emulate Bézier subdivision by inserting d knots at a time. (Lane/Risenfeld, Rockwood, Schneider).
- “Preconditioning”, insert knots from neighboring rays.
- Estimate root multiplicity and detect roots of n'th derivative. (Strictly alternating control polygon.)
- Detect critical points, use as start value to search for singularities.

Singularity detection

1. For misses, find smallest absolute value along ray, w_0.
2. Flag as singularity if:

$$|g(p/m, q/n, w_0)| + \| \nabla g(p/m, q/n, w_0) \| < \epsilon.$$

- How to determine ϵ?
- Vulnerable to scaling.

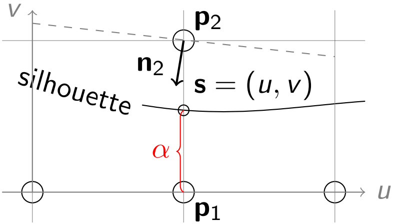


$x^2 - y^3 = 0.$

Antialiasing

Due to discrete sampling, aliasing effects will occur.

- Suppose neighboring pixels $\mathbf{p}_1$, $\mathbf{p}_2$ differ.
  - I.e. $\nabla\mathbf{p}_1 \cdot \nabla\mathbf{p}_2 < \epsilon$.
- We seek a point $\mathbf{s}$ on the separating curve between $\mathbf{p}_1$ and $\mathbf{p}_2$.
- $color(\mathbf{p}_1) :=$
  $(1.0 - \alpha)\ color(\mathbf{p}_1) + \alpha\ color(\mathbf{p}_2)$



Antialiasing II

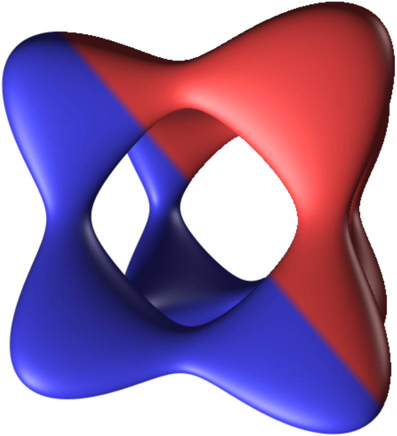
At silhouettes

$$g(\mathbf{s}) = 0 \text{ and } g_w(\mathbf{s}) = 0.$$

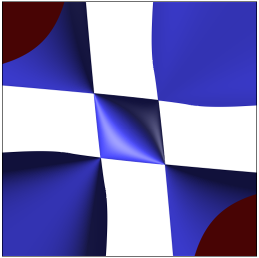
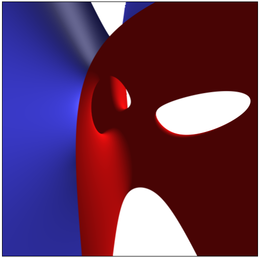
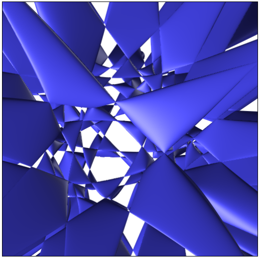
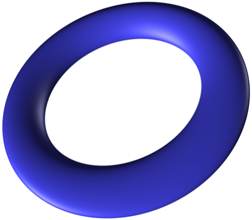
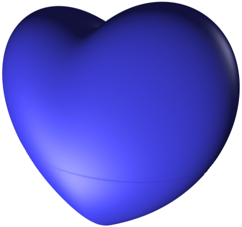
- Use Newton methods on

$$\mathbf{h}(v, w) := (g(\mathbf{s}), g_w(\mathbf{s})) = (0, 0).$$

- Restrict to plane between $\mathbf{p}_1, \mathbf{p}_2$.
- If leaving domain, search for $g(\mathbf{s}) = 0$.



Gallery and performance


Future work

- Interval spline methods:
  - Topological correctness.
  - Empty-space skipping.
- Bounding box calculations
- Efficient data structures for splines

Thank you for listening

Questions?

Contact info

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- Tel: +47 97 18 16 14

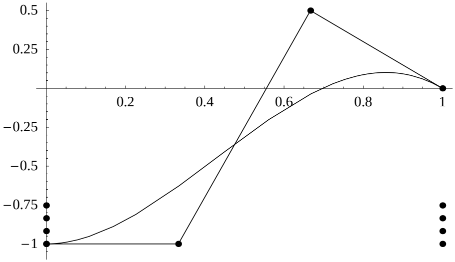
Mørken and Reimers
An unconditionally convergent method for computing zeros of splines and polynomials
Math. of Comp. 76, 2006

Zero Algorithm [MrkenReimers 2007]

Idea: Repeated knot insertion at zeros of $F_{\mathbf{t}}$

Repeat for $j = 0, 1, \cdots$ until convergence or $F_{\mathbf{t}^j}$ has no zeros

1. Find the smallest value x_{j+1} such that $F_{\mathbf{t}^j}(x_{j+1}) = 0$ or stop
2. Let $\mathbf{t}^{j+1} = \mathbf{t}^j \cup \{x_{j+1}\}$ and form $F_{\mathbf{t}^{j+1}}$



Error $|x_j - z|$: 1.41e-1

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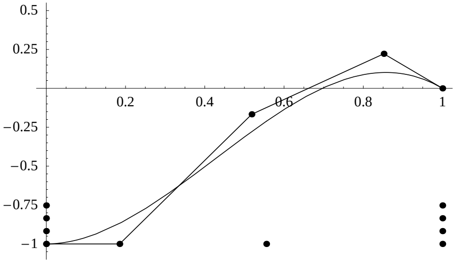

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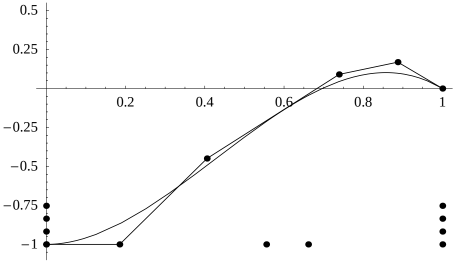

Error $|x_j - z|$: 1.41e-1 3.48e-2 1.31e-2

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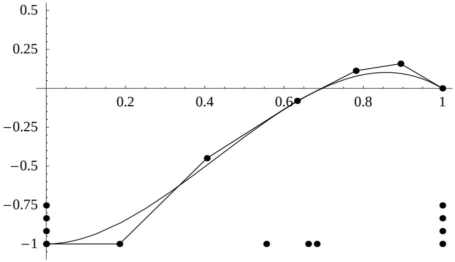

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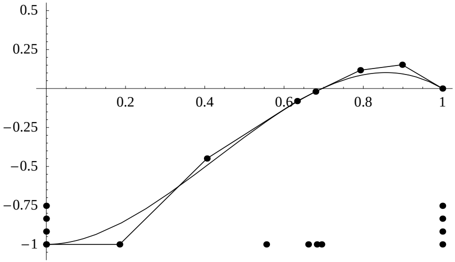

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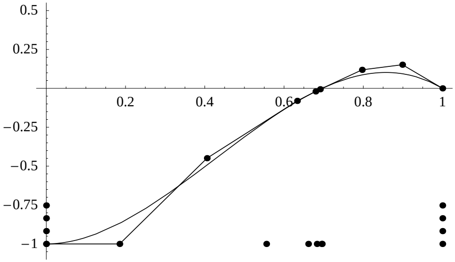

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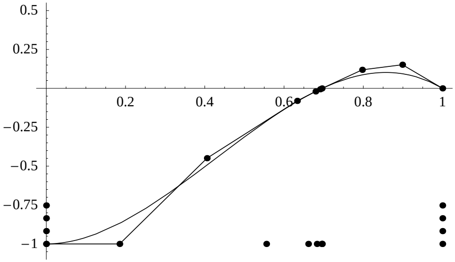

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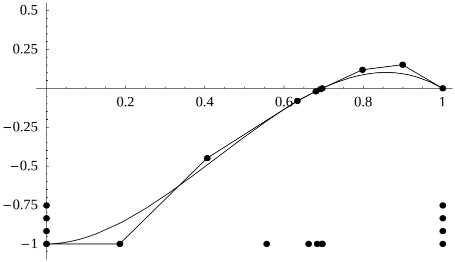

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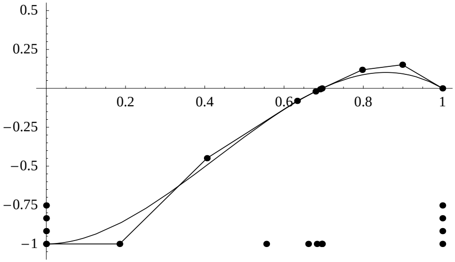

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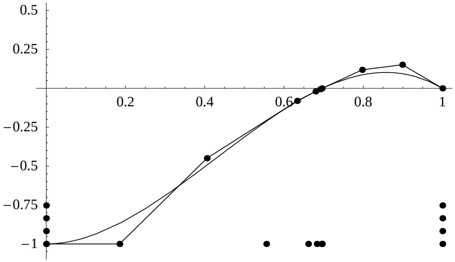

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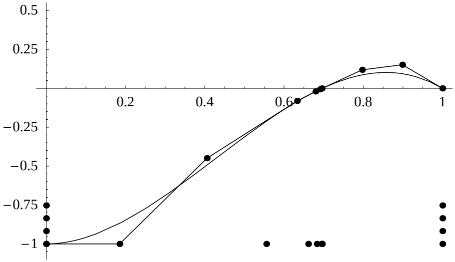

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