NBI

REPRINT 159

NORWEGIAN BUILDING RESEARCH INSTITUTE

REPRINT FROM

BIT

BIND 8
HEFTE NR. 2
1968

BIT 8 (1968), 86–103

A METHOD FOR INTEGRATION OF UNSTABLE SYSTEMS OF ORDINARY DIFFERENTIAL EQUATION SUBJECT TO TWO-POINT BOUNDARY CONDITIONS

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Abstract.

Instability problems in systems of differential equations are discussed. A matrix technique is given for producing numerical solutions to a system of ordinary differential equations with boundary conditions specified at each end of the interval when the system contains dominant solutions which give rise to numerical instability in conventional integration methods. A method of "bringing up the initial conditions" is described, whereby the two-point nature of the problem is made use of to stabilize the system. Three numerical examples are included.

Introduction.

We are considering a system of N linear equations of the form

$$\frac{d}{dx}F(x) = A(x)F(x)+B(x) \tag{1.1}$$

subject to the boundary conditions

$$\begin{aligned} J_0F(x_0) &= C_0 \\ J_nF(x_n) &= C_n \end{aligned} \tag{1.2}$$

where: F is a vector (N) of the dependent variables
A is an $(N\times N)$ matrix, assumed real and nonsingular
B an (N) vector
J_0 an $((N-M)\times N)$ matrix and J_n an $(M\times N)$ matrix
C_0 and $(N-M)$ vector and C_n an (M) vector

The indices 0 and and n refer to the beginning and end points of the considered interval, see Fig. 1.

The above type of equations frequently turns up in the stress analysis of elastic bodies where the partial differential equations in two or three variables can be reduced by various methods, (e.g. Fourier Analysis), to a set of ordinary differential equations of the above type, [3].

The A-matrix will then be a function of the geometry and material characteristics of the structure, while the vector B will be a function of the imposed loads. The function vector F contains the displacements and their derivatives.

The boundary value problem in ordinary differential equations has been treated by a number of writers. The monograph by Fox [5] gives a detailed treatment of the subject based upon finite-difference techniques. Schemes suggested by other writers are mostly based on the idea of determining the missing initial values, by direct or iterative methods, so that the conditions specified at the far end of the interval will be satisfied, whereafter the problem becomes one of the initial type, i.e. a problem of direct integration. This technique will not always work, even if all the initial values are known exactly, as is demonstrated in the second example of sec. 4.

A paper by Midgley [6] is devoted to this particular aspect of the initial-value problem with dominant solutions, a method being given for the calculation of the subdominant complementary functions.

An important paper by Conte [10] presents a modification of a method proposed by Godunov [11]. The basic idea consists in orthonormalizing the complementary function vectors (T in eq. 2.5) at intermediate points in the interval, before the linear dependence of these vectors has become too pronounced.

The method presented in sec. 2 of the present paper is somewhat related to the Godunov–Conte method, but arrived at independently and from a different basis [3].

An interesting embedding technique is presented by Bellman et al. [9]. Here the unstable boundary value problem is transformed into a nonlinear but stable initial-value problem by introducing the interval length s as a new independent variable.

Extensions of methods to nonlinear problems have been proposed [7, 8] in the form of iterations upon a linear system. It will be appreciated that methods for linear systems that are sufficiently general and robust with regard to numerical instability can be made good use of in this field of applied analysis.

In the present paper, it is assumed that the elements of A are sectionally continuous in the interval (a,b) but not necessarily expressible analytically. This will be so when the equations are established in the form

$$CF' = A_1F + B_1$$

where C can be inverted only numerically to yield the system (1.1).

A formal solution for the interval $(0,x)$ can be established in the form (see [1, 2, 4])

$$F(x) = \sum_{k=0}^{\infty} G_k(x)F_0 + \sum_{k=1}^{\infty} L_k(x) = G(x)F_0 + L(x) \tag{1.3}$$

where $F_0 = F(x_0)$ and

$$G_0(x) = I,\ G_{k+1}(x) = \int_0^x A(u)G_k(u)du$$

$$L_1(x) = \int_0^x B(u)du, \quad L_{k+1}(x) = \int_0^x A(u)L_k(u)du\ .$$

The transfer matrix $G(x)$ is termed the matrizant. For $A = \text{const.}$ it reduces to the matrix exponential

$$G(x) = I + Ax/1! + (Ax)^2/2! + \ldots = e^{Ax} \tag{1.4}$$

The series can be shown always to be convergent [1], although the convergence may be so slow that the direct evaluation by (1.4) may be practically impossible, an aspect which we shall consider in section 3.

In the important case when A and B are constant, (1.3) can be written as

$$F_b = e^{As}F_a + A^{-1}(e^{As} - I)B \tag{1.5}$$

where $s = b - a$.

Inversion of A can be avoided by expanding the last term of (1.5) as

$$L = (sI/1 + s^2A/2! + \ldots)B \tag{1.6}$$

In general, we want to determine the function vectors, $F_0, F_1, \ldots F_n$ at a number of points in the interval, as indicated in Fig. 1.

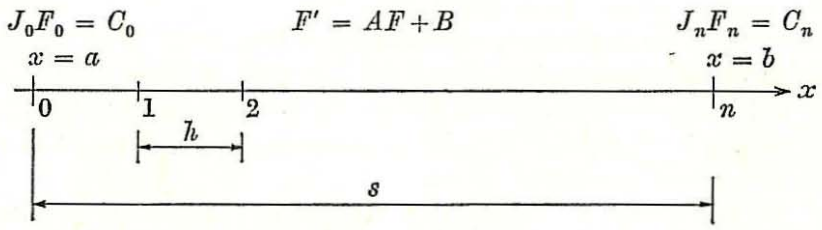

Fig. 1

A formally correct way of doing this would be first, by some suitable method, to integrate (1.1), so that we get the G and L matrices for the entire interval

$$F_n = GF_0 + L \tag{1.7}$$

and then use the boundary conditions (1.2) to determine the startpoint vector F_0, by solving the system

$$\left[\frac{J_0}{J_n G}\right] F_0 = \left[\frac{C_0}{C_n - J_n L}\right] \tag{1.8}$$

Having found F_0 we have transformed the problem to an initial-value problem and we can apply our integration technique to determine F at as many intermediate points in the interval as we wish, using the known vector F_i to determine the next vector F_{i+1}.

In the following we shall consider two special problems which may arise when we try to produce numerical solutions to the system (1.1) and (1.2). Firstly, there is the problem of numerical instability, i.e. the roundoff errors during the integration propagate to an extent which makes the results unacceptable. To this effect may be added the effects of the system (1.8) being ill-conditioned, yielding inaccurate values for the initial vector F_0 which makes the situation even worse.

Secondly we shall briefly deal with the problem of computing the transfer matrices G and L as defined by (1.3) when the series (1.4) and (1.6) are too slowly convergent to be of practical use.

The key to an understanding of both problems is the eigenvalue spectrum of A, which can be represented as on Fig. 2. As we shall see, the problem of convergence of the series (1.4) (or of integration methods based upon a Taylor-type expansion, like Runge–Kutta methods) arise when the spectrum has a large upper bound $|\lambda|_{\max}$, while instability and ill-conditioning is chiefly a consequence of a large spectral width μ, measured along the real axis, both causes being amplified by the total length s of the interval over which the equations are to be integrated.

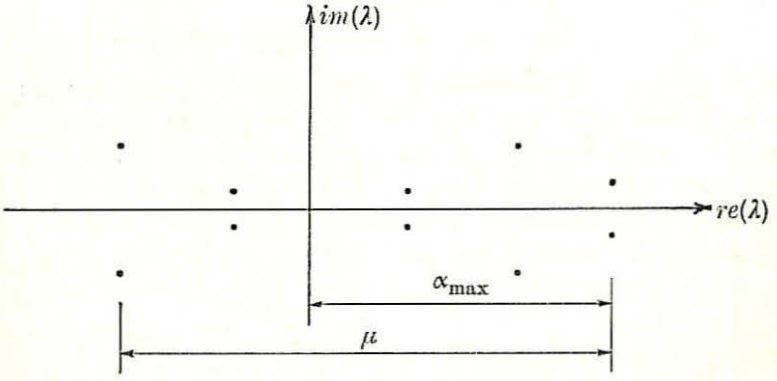

Fig. 2. The eigenvalue spectrum of A.

In order to see how the problems of instability and ill-conditioning arise, let us for the sake of simplicity assume that A is constant and write A in terms of its eigenvalues and eigenvectors:

$$A = Y\Lambda Y^{-1} \tag{1.9}$$

where Y is the augmented $(N \times N)$ matrix of eigenvectors (y_i), assumed non-singular, and Λ is the diagonal matrix of eigenvalues λ_i. Considering only the homogeneous part of (1.1) we see that integration over the interval s yields

$$F_n = GF_0 = Ye^{\Lambda s}Y^{-1}F_0 = YDY^{-1}F_0 \tag{1.10}$$

The eigenvalues, d, of G have the form

$$d = e^{\lambda s} = e^{\alpha s}(\cos\beta s + i \sin\beta s) \tag{1.11}$$

where $\lambda = \alpha + i\beta$.

In the complex plane, for varying s, the loci of d will be spirals as shown on Fig. 3.

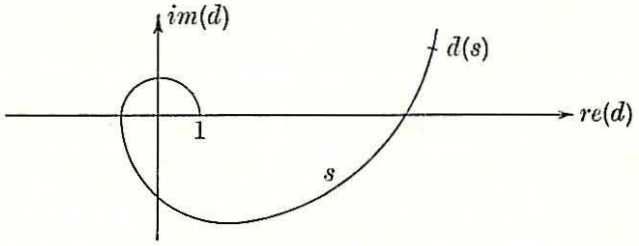

Fig. 3. Growth of the eigenvalues of G.

The ratio between the largest and the smallest modulus in the eigenvalue spectrum of G can be represented logarithmically by

$$\eta = \log_{10}(r_{\max}/r_{\min}) = \mu \cdot s \cdot \log_{10} e = 0.4343\mu \cdot s \tag{1.12}$$

η can be used as an indicator of the instability of (1.1) as well as a measure of the degree of singularity of G. As η approaches the number of decimal digits with which the machine works, the smallest pair of eigenvalues of G will for practical purposes approach zero, rendering G singular, its nullity depending upon the number of vanishing eigenvalues. As s goes on increasing, the largest pair of eigenvalues will become increasingly dominant, and we may in fact reach a stage where G, as represented in the machine, has the rank 2.

Clearly, it is possible before this extreme stage is reached, that the system (1.8) becomes so ill-conditioned that an acceptable solution is

unattainable. This will to some extent depend upon the form of J_0 and J_n.

On the other hand, from (1.10) it would seem that the largest relative error in any element of F_i depends rather upon the width μ of the spectrum and the interval $s_i = x_i - a$, if we assume that the smallest element of F grows exponentically with $\alpha_{\min} \cdot s$ while the absolute error due to roundoff grows exponentically with $\alpha_{\max} \cdot s$. We may therefore adopt the quantity $\eta_i = 0.4343\mu \cdot s_i$ as an indicator of the sensitivity of the system to roundoff errors.

Clearly then, for a sufficiently large interval s, the propagating effects of roundoff errors can become of the order of magnitude of the elements of F themselves, with disastrous results.

In practice, for an 8th-order system (1.1), say, it probably suffices for η to have a value of 3 or 4 to give rise to a degree of illconditioning which renders the system (1.8) too sensitive to roundoff errors to allow the sequence of vectors F_i to be computed with sufficient accuracy. We may find that having determined a value of the starting vector F_0, and integrated this up to the end point, the boundary conditions (1.2b) at the end point will not be satisfied with acceptable accuracy.

In what follows, we shall present a procedure whereby the total interval s is split into a few sub-intervals of length h, and an inversion is performed after each sub-interval has been integrated, whereby at each stage, i, F_i is expressed by M linear combinations of its own elements.

2. Stepwise inversion and the "bringing up" of initial conditions.

From the vector F we form two subvectors $\bar{F}$ and $\underline{F}$ of order M and $N-M$, respectively, by premultiplying F by the operators $\bar{Q}$ and $\underline{Q}$

$$\begin{aligned} \bar{F} &= \bar{Q}F \\ \underline{F} &= \underline{Q}F \end{aligned} \tag{2.1}$$

where $\bar{Q}$ is an $(M \times N)$ matrix and $\underline{Q}$ is an $((N-M) \times N)$ matrix, to be chosen (see below).

Then

$$\begin{aligned} F &= \begin{bmatrix} \bar{Q} \\ \cdots \\ \underline{Q} \end{bmatrix}^{-1} \begin{bmatrix} \bar{F} \\ \cdots \\ \underline{F} \end{bmatrix} = [\bar{R} \vdots \underline{R}] \begin{bmatrix} \bar{F} \\ \cdots \\ \underline{F} \end{bmatrix} \\ F &= \bar{R}\bar{F} + \underline{R}\underline{F} \end{aligned} \tag{2.2}$$

The boundary conditions (1.2a) give

$$J_0(\bar{R}\bar{F}_0 + \underline{R}\underline{F}_0) = C_0$$

whence

$$\underline{F}_0 = (J_0\underline{R})^{-1}(C_0 - J_0\bar{R}\bar{F}_0)$$

(2.2) gives

$$F_0 = \bar{R}\bar{F}_0 + \underline{R}(J_0\underline{R})^{-1}(C_0 - J_0\bar{R}\bar{F}_0)$$

which can be arranged thus

$$F_0 = K_0\bar{F}_0 + V_0 \tag{2.3}$$

where K is an $(N \times M)$ matrix and V is an (N) vector.

We have now succeeded in expressing, at the starting point, the full vector F_0 by M linear combinations, $\bar{F}_0$, of F_0. Below we shall show how we, after integrating (1.1) over a step, can obtain an identical representation to (2.3) at any point i.

Let us assume that we possess a suitable integration method for the numerical evaluation of the matrix G and the vector L in (1.3). Integrating over the step $(0,h)$, we then obtain by (1.3)

$$F_1 = GF_0 + L \tag{2.4}$$

using (2.3) we get

$$F_1 = G(K_0\bar{F}_0 + V_0) + L = T\bar{F}_0 + U \tag{2.5}$$

where T is an $(N \times M)$ matrix and U is an (N) vector.

Using (2.1)

$$\bar{F}_1 = \bar{Q}T\bar{F}_0 + \bar{Q}U = \bar{T}\bar{F}_0 + \bar{U}$$

yielding

$$\bar{F}_0 = \bar{T}^{-1}(\bar{F}_1 - \bar{U}) \tag{2.6}$$

By (2.5) we can now express F_1 in terms of $\bar{F}_1$

$$F_1 = T\bar{T}^{-1}(\bar{F}_1 - \bar{U}) + U$$

or

$$F_1 = K_1\bar{F}_1 + V_1 \tag{2.7}$$

The form of (2.7) is identical to the form of (2.3) at the starting point, and we can therefore say that we have "brought up" the initial conditions. The form (2.3) or (2.7) is referred to as the "point form".

We proceed, as above, step by step, until we reach the end-point of the interval where we have

$$F_n = K_n\bar{F}_n + V_n \tag{2.8}$$

and by using the boundary conditions (1.2b) we find

$$\bar{F}_n = (J_nK_n)^{-1}(C_n - J_nV_n) \tag{2.9}$$

whence F_n is found by (2.8).

Having determined the function vector F at the end point n, we can now compute the succession of vectors $F_{n-1}, F_{n-2} \ldots F_0$ by making use of the recurrence relation

$$F_{i-1} = K_{i-1}\bar{T}^{-1}(\bar{F}_i - \bar{U}) + V_{i-1}$$

or

$$F_{i-1} = H_{i-1}\bar{F}_i + W_{i-1} \tag{2.10}$$

which can be established on the basis of (2.6) and (2.3).

The $(N \times M)$ matrices H and the (N) vectors W can readily be formed during the forward integration and stored.

By the above method the tendency of the roundoff errors to grow exponentially over the length of integration is checked, at intermediate stages, by the transformation to the "point form" (2.7), of the integrated equation (2.4). A rigorous analysis of the numerical mechanism involved, with estimates for errors etc., is outside the scope of this paper.

A word may be said about the choice of $\bar{Q}$ and $\underline{Q}$. Clearly, in order for F_0 to be uniquely determined in terms of $\bar{F}_0$ and C_0 as well as in terms of $\bar{F}_0$ and $\underline{F}_0$ the matrices

$$\begin{bmatrix} J_0 \\ \cdots \\ \bar{Q} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{Q} \\ \cdots \\ \underline{Q} \end{bmatrix} \quad \text{must be nonsingular .}$$

In problems of elasticity, where N always is an even number ($N = 4, 6, 8 \ldots$) and $M = N/2$, it is always possible to specify $\bar{Q}$ and $\underline{Q}$ so that the above conditions are fulfilled for any set of physically admittable boundary conditions. By letting $\bar{F}$ and $\underline{F}$ consist of combinations of the form $\bar{f}_i = u_i + \sigma_i$ and $\underline{f}_i = u_i - \sigma_i$ where u is a displacement and σ the corresponding stress, this is achieved.

3. The evaluation of the transfer matrices G and L when the eigenvalues of A are large.

In general, the elements of A and B vary with x, and we must therefore resort to numerical approximation methods for the calculation of G and L. We shall briefly indicate a method of integration which is easily coded and which is particularly efficient when the eigenvalues of A are of an order of magnitude such as to make the series evaluation (1.4), the convergence of which can be estimated by (1.10), practically impossible for a feasible step length h which otherwise would be small enough to take proper care of the variable nature of the coefficients of the equation.

From (1.10) it follows that the number of terms of the expansion (1.4) that would be required to obtain a certain accuracy depends on the

product $h \cdot \lambda_{\max}$. If we now consider a fourth-order Runge–Kutta method, which in the case $A = \text{const.}$ simply is a Taylor expansion truncated after the fifth term it is easy to demonstrate that for five figure accuracy

$$h \cdot \lambda_{\max} \leqq 0.35 \text{ approximately} \tag{3.1}$$

In practice, therefore, depending upon the largest eigenvalues of A, it may not be possible to employ a Runge–Kutta method without reducing the step-size h far below what is practical.

The matrizant evaluation (1.3) is generally not suited for numerical computation. Pipes [12] suggests a perturbation method whereby G is computed, for the interval $(0,h)$ as

$$G = e^{\bar{A}h}(1+\delta) \tag{a}$$

where

$$\delta = \delta_1 + \delta_2 + \ldots \tag{b}$$

$$\bar{A} = \frac{1}{h}\int_0^h A(u)\,du$$

$$\delta_1(x) = \int_0^x e^{-\bar{A}u} a(u) e^{\bar{A}u}\,du$$

$$\delta_{k+1}(x) = \int_0^x e^{-\bar{A}u} a(u) \delta_k(u)\,du$$

$$a(u) = A(u) - \bar{A}$$

stating that if the perturbation matrix $a(u)$ is small in the interval, δ can be computed taking only one or two terms. However, the convergence of the series (b) depends on the eigenvalues of $\bar{A}$ as well, and although δ in itself may be quite small, the series (b) may be illbehaved and converge slowly when A has large eigenvalues. While the form (a) is a very desirable basis for numerical computation, a practical way of computing δ, for the types of equations studied in the present paper, is still lacking and we make the approximation $\delta = 0$.

We then take for the interval h

$$\begin{aligned} \bar{A} &= (A_i + A_{i+1})/2 \\ \bar{B} &= (B_i + B_{i+1})/2 \end{aligned} \tag{3.2}$$

or, eventually, evaluate A and B at an intermediate point and use Simpson's rule.

We then evaluate G and L according to (1.5) and in evaluating the matrix exponential make use of the relation

$$e^{\bar{A}h} = (e^{\bar{A}h/k})^k \tag{3.3}$$

where

$$k = 2^m$$

We compute (3.3) as

$$(\cdots((e^{\bar{A}h/k})^2)^2\ldots)^2 \tag{3.4}$$

By suitable choice of m, say $m=6$, we can evaluate the innermost paranthesis with a few terms in the expansion (1.4), then square this matrix, then square the result again and so on, m times. L can then be evaluated as indicated by (1.5), whereby an inversion of $\bar{A}$ has to be performed.

4. Numerical examples.

4.1. *Example 1.*

Consider the equation

$$y'''' - 4y''' + 6y'' - 4y' + 5y = 1 \tag{4.1}$$

$$\begin{aligned} y &= y' = 0, \ x = 0 \\ y'' &= y''' = 0, \ x = s \end{aligned} \tag{4.2}$$

The equivalent 1st order system is

$$\frac{d}{dx}\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ -5 & 4 & -6 & 4 \end{bmatrix}\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} + \begin{bmatrix} . \\ . \\ . \\ 1 \end{bmatrix}$$

or

$$F' = AF + B \tag{4.3}$$

and the boundary conditions

$$\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \end{bmatrix} F(o) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } J_0F_0 = C_0$$

$$\begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \end{bmatrix} F(s) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad J_nF_n = C_n \tag{4.4}$$

The eigenvalues of A are

$$\lambda_{1,2} = \pm i, \ \lambda_{3,4} = 2 \pm i$$

hence

$$\mu = 2$$

The system (4.3) and (4.4) has been solved numerically by stepwise inversion as well as by the "classical" method described in section 1, involving inversion over the whole interval s.

The linear combinations $\overline{F}$ are simply

$$\overline{F} = \begin{bmatrix} y'' \\ y''' \end{bmatrix} \text{ so that } \overline{Q} = \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \text{ and } \underline{Q} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \tag{4.5}$$

Complete solutions have been produced for different ranges of the variable x, thus $s = 2.0, 4.0 \ldots 18.0$.

For $s = 2.0$ the results of the two methods correspond to 5 decimal places, for $s = 4.0$ to 3 places.

For $s = 6.0$ we have $\eta = 5.2$, while for $s = 8.0$ we get $\eta = 6.9$, and the situation is clearly critical.

The results for the two latter interval lengths are given in table 4.1, illustrating the instability of the system in the critical range of s. For the method of stepwise inversion the total execution time including printout was 1.15 sec. on a UNIVAC 1107.

The fact that A is constant was not made use of in writing the programme; in fact, no special time-saving features were incorporated. Execution time could probably be halved by paying more attention to such details.

4.2. *Example 2.*

The equations

$$\begin{aligned} v'' &= \alpha(v-u),\ u(o) = v'(o) = 0 \\ u'' &= \beta(u-v),\ u'(s) = 0,\ v'(s) = c \end{aligned} \tag{4.6}$$

have the analytical solution

$$\begin{aligned} u &= c[\beta\gamma/r^3 + \beta x/r^2 - \beta\gamma \cosh(rx)/r^3 + \beta^2 \sinh(rx)/\alpha r^3] \\ v &= c[\beta\gamma/r^3 + \beta x/r^2 + \alpha\gamma \cosh(rx)/r^3 - \beta \sinh(rx)/r^3] \end{aligned}$$

where

$$\begin{aligned} r &= \sqrt{\alpha+\beta} \\ \gamma &= (\beta/\alpha \cosh(rs) + 1)/\sinh(rs) \end{aligned} \tag{4.7}$$

In order to isolate the effect of roundoff errors we can compare the solution obtained by numerical integration from the exact initial values with the analytical solution. For $\alpha = \beta = 0.25$ and $s = 10.0$ the solutions correspond to 5 decimal places. For $\alpha = \beta = 2.5$ and $s = 10.0$ the errors

are of the order of the elements themselves towards the end of the interval, as can be seen by table 4.2. The method of stepwise inversion, however, is completely stable for both sets of parameters. 10 steps were used, $\bar{Q}$ being taken as

$$\bar{Q} = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

and $c = 10^{-3}$.

4.3. *Example 3.*

The stress-displacement analysis of elastic shells in the form of a surface of revolution subjected to a load which can be expressed as a Fourier series around the circumference, leads to an 8th order system where the function vector $\{Y_1 Y_2 \ldots Y_8\}$ contains the displacements u, v and w and certain of their derivatives. Internal stresses in the shell are found by premultiplying Y by a "stiffness matrix" H. However, this need not concern us here, and we only consider the differential equation. The A and B matrices have been printed out (see table 4.3) at two adjacent stations in the middle of the interval. As can be seen, there is some variation in A. B happens to be constant, as we have considered a radial load with constant amplitude over the height of the shell, which is of the form of a hyperboloid fixed at the base and free at the top, where it is loaded with an additional load in the s-direction of $100 \cdot \cos\theta$.

20 subdivisions were used. The results at every second point are given in table 4.3. The G and L matrices were computed as described in section 3 and 12 terms were used in the expansion, (1.4), and m (eq. 3.3) was 6.

The accuracy of the solution can be estimated by checking the over-all equilibrium of the shell, i.e. comparing the total external load in various directions, overturning moments, etc. with the reactions at the base. Such a check gives an accuracy of about 1%, part of which is due to other causes than the solution of the differential equations, and which at any rate is adequate for engineering purposes. More steps would improve the accuracy.

5. Computation scheme.

Below is given an outline of a computation scheme, based on the author's FORTRAN program, for the described method.

Input of parameters for problem
form $\bar{Q}$ and $\underline{Q}$
form J_0 and C_0

transform $J_0F_0=C_0$ to $F_0=K_0\overline{F}_0+V_0$ (2.1-3)
$x=a$
$dx=(b-a)/n$ (n=number of intervals in (a,b))
form A_1 and B_1 (at $x=a$)
DO 1 $i=1,n$
$x=x+dx$
form A_2 and B_2 (at $x=x_i$)
form $\overline{A}=(A_1+A_2)/2$ and $\overline{B}=(B_1+B_2)/2$
Integrate to form G and L (method of sec. 3, Runge–Kutta, or other)
form $T=GK_{i-1}$ and $U=GV_{i-1}+L$ (2.5)
compute $(\overline{Q}T)^{-1}=\overline{T}^{-1}$ (2.6)
form $K_i=T\overline{T}^{-1}$ and $V_i=U-K_i\overline{U}$ (2.7)
form $H_{i-1}=K_{i-1}\overline{T}^{-1}$ and $W_{i-1}=V_{i-1}-K_{i-1}\overline{T}^{-1}\overline{U}$ (2.10)
store H_{i-1} and W_{i-1}
copy A_2 and B_2 into A_1 and B_1
1 CONTINUE
form J_n and C_n
solve for $\overline{F}_n=(J_nK_n)^{-1}(C_n-J_nV_n)$ (2.9)
compute $F_n=K_n\overline{F}_n+V_n$
print (store) F_n
DO 2 $i=n,\ 1,\ -1$
compute $F_{i-1}=H_{i-1}\overline{F}_i+W_{i-1}$ (2.10)
print (store) F_{i-1}
2 CONTINUE
STOP
END

6. Tables.

Stepwise inversion
$s = 6.00$

pt. no.	y	y'	y''	y'''
0	.00000000	.00000000	.19999883	.00000017
1	.03493289	.11292824	.16506865	−.11291641
2	.12752949	.18641248	.07249118	−.18635079
3	.24545095	.19480174	−.04535606	−.19459621
4	.34753194	.13521637	−.14724220	−.13477349
5	.39817975	.02853871	−.19764160	−.02838208
6	.37976027	−.08815765	−.18000939	.08412815
7	.29828883	−.17590268	−.10559110	.15208700
8	.17928293	−.21176738	−.01656034	.12566026
9	.05280673	−.20575656	.02338871	−.00362583
10	−.06758755	−.19833209	.00000000	.00000000

Inversion for whole interval
$s = 6.00$

pt. no.	y	y'	y''	y'''
0	.00000000	.00000000	.19999883	.00000017
1	.03493289	.11292824	.16506866	−.11291635
2	.12752949	.18641249	.07249130	−.18635044
3	.24545100	.19480193	−.04535551	−.19459504
4	.34753225	.13521718	−.14724055	−.13477099
5	.39818095	.02854097	−.19763857	−.02838129
6	.37976328	−.08815426	−.18001095	.08410498
7	.29829212	−.17590901	−.10563295	.15195128
8	.17926774	−.21183942	−.01677253	.12517175
9	.05268701	−.20608164	.02268704	−.00480711
10	−.06807613	−.19931944	−.00150644	−.00108808

Table 4.1a.

Stepwise inversion
$s = 8.00$

pt. no.	y	y'	y''	y'''
0	.00000000	.00000000	.19999820	−.00000323
1	.06066208	.14346862	.13933628	−.14347149
2	.20584388	.19990771	−.00584514	−.19990969
3	.34748023	.13508270	−.14747971	−.13508032
4	.39965792	−.01168118	−.19964797	.01170085
5	.33073331	−.15134865	−.13069588	.15138515
6	.18252884	−.19920325	.01746822	.19892904
7	.04488037	−.12649821	.15416027	.12365916
8	.00051459	.02081438	.19290856	−.03374180
9	.07257974	.14750353	.10553210	−.16713786
10	.21024968	.18157298	.00000000	.00000000

Inversion for whole interval
$s = 8.00$

pt. no.	y	y'	y''	y'''
0	.00000000	.00000000	.19999820	−.00000323
1	.06066208	.14346861	.13933632	−.14347133
2	.20584390	.19990782	−.00584471	−.19990864
3	.34748062	.13508375	−.14747754	−.13507682
4	.39966018	−.01167723	−.19964348	.01169896
5	.33073905	−.15134723	−.13071904	.15128526
6	.18251278	−.19929081	.01719804	.19828621
7	.04462775	−.12713802	.15286401	.12167343
8	−.00083536	.01854766	.19059150	−.03167602
9	.06946055	.14755680	.12134153	−.10416654
10	.22184106	.23755038	.16595310	.38392383

Table 4.1b.

$\alpha = 2.50 \quad \beta = 2.50 \quad s = 10.0$

Analytical solution

pt. no.	u	v	u'	v'
0	−0.00000	4.47214−04	1.00000−03	−0.00000
1	6.99708−04	7.47505−04	5.53439−04	4.46561−04
2	1.22105−03	1.22616−03	5.05711−04	4.94289−04
3	1.72333−03	1.72388−03	5.00610−04	4.99390−04
4	2.22358−03	2.22363−03	5.00060−04	4.99940−04
5	2.72360−03	2.72361−03	4.99993−04	5.00007−04
6	3.22358−03	3.22364−03	4.99935−04	5.00065−04
7	3.72333−03	3.72388−03	4.99390−04	5.00610−04
8	4.22105−03	4.22616−03	4.94289−04	5.05711−04
9	4.69971−03	4.74751−03	4.46561−04	5.53439−04
10	5.00000−03	5.44721−03	−0.00000	1.00000−03

Stepwise inversion

pt. no.	u	v	u'	v'
0	0.00000	4.47214−04	1.00000−03	0.00000
1	6.99708−04	7.47505−04	5.53439−04	4.46561−04
2	1.22105−03	1.22616−03	5.05712−04	4.94289−04
3	1.72333−03	1.72388−03	5.00610−04	4.99390−04
4	2.22358−03	2.22364−03	5.00065−04	4.99936−04
5	2.72360−03	2.72361−03	5.00000−04	5.00000−04
6	3.22358−03	3.22364−03	4.99935−04	5.00065−04
7	3.72333−03	3.72388−03	4.99390−04	5.00610−04
8	4.22105−03	4.22616−03	4.94289−04	5.05711−04
9	4.69971−03	4.74751−03	4.46561−04	5.53439−04
10	5.00000−03	5.44721−03	−1.73472−18	1.00000−03

Integration from exact initial values

pt. no.	u	v	u'	v'
0	−0.00000	4.47214−04	1.00000−03	−0.00000
1	6.99708−04	7.47505−04	5.53439−04	4.46561−04
2	1.22105−03	1.22616−03	5.05711−04	4.94289−04
3	1.72333−03	1.72388−03	5.00609−04	4.99391−04
4	2.22357−03	2.22364−03	5.00053−04	4.99947−04
5	2.72355−03	2.72366−03	4.99891−04	5.00109−04
6	3.22312−03	3.22409−03	4.98918−04	5.01082−04
7	3.71908−03	3.72814−03	4.89871−04	5.10129−04
8	4.18122−03	4.26599−03	4.05227−04	5.94773−04
9	4.32704−03	5.12017−03	−3.86744−04	1.38674−03
10	1.51317−03	8.93404−03	−7.79679−03	8.79679−03

Table 4.2.

Stress analysis of hyperboloid with radial load = 1·cos(*theta*)

*J*0 and *C*0 matrices

1.000+00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	1.000+00	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	1.000+00	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	1.000+00	0.000	0.000	0.000

A and *B* matrices at $x = 41.81$

−0.000	−0.000	−0.000	1.000+00	−0.000	−0.000	−0.000	−0.000	0.000
−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	0.000
−0.000	−0.000	−0.000	−0.000	−0.000	1.000+00	−0.000	−0.000	0.000
3.652−03	1.250−03	3.281−03	9.607−03	−5.575−02	2.670−08	−2.780−06	−0.000	0.000
4.998−04	6.235−04	2.810−04	2.230−02	9.607−03	4.219−03	−0.000	−1.137−08	0.000
−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	1.000+00	−0.000	0.000
0.000	0.000	2.654−05	0.000	0.000	1.452−03	9.607−03	−4.608−03	−0.000
3.417+02	8.834+01	3.269+02	0.000	−1.099+03	2.880−03	−2.997−01	9.607−03	1.000+00

A and *B* matrices at $x = 52.07$

−0.000	−0.000	−0.000	1.000+00	−0.000	−0.000	−0.000	−0.000	0.000
−0.000	−0.000	−0.000	−0.000	1.000+00	−0.000	−0.000	−0.000	0.000
−0.000	−0.000	−0.000	−0.000	−0.000	1.000+00	−0.000	−0.000	0.000
4.441−03	1.313−03	4.005−03	9.158−03	−6.144−02	3.120−08	−3.407−06	−0.000	0.000
5.252−04	7.225−04	2.796−04	2.458−02	9.158−03	3.912−03	−0.000	−1.564−08	0.000
−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	1.000+00	−0.000	0.000
0.000	0.000	3.073−05	0.000	0.000	1.729−03	9.158−03	−4.608−03	−0.000
4.172+02	9.328+01	4.025+02	0.000	−1.019+03	3.335−03	−3.641−01	9.158−03	1.000+00

J_n and C_n matrices

7.106+02	−4.836+03	0.000	−1.042+05	0.000	0.000	0.000	0.000	0.000
2.418+03	−3.553+02	7.778+02	0.000	−2.604+05	0.000	0.000	0.000	1.000+02
0.000	0.000	−9.354−02	0.000	0.000	2.961−01	2.170+02	0.000	0.000
0.000	0.000	−5.498−02	0.000	0.000	8.060+00	0.000	1.000+00	0.000

Numerical solution

x	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
102.38	7.999−02	−2.238−02	−8.597−02	1.585−03	1.324−04	−1.565−03	−3.492−05	7.884−03
92.31	6.480−02	−2.349−02	−6.893−02	1.427−03	8.392−05	−1.953−03	−2.210−05	−1.969−01
82.30	5.144−02	−2.384−02	−5.343−02	1.235−03	−1.976−05	−1.661−03	−1.964−05	−1.642−01
72.29	4.010−02	−2.299−02	−4.058−02	1.027−03	−1.462−04	−1.295−03	−1.831−05	−1.052−01
62.23	3.074−02	−2.096−02	−3.051−02	8.346−04	−2.502−04	−9.297−04	−1.677−05	−4.530−02
52.07	2.309−02	−1.807−02	−2.301−02	6.745−04	−3.133−04	−6.250−04	−1.449−05	−1.188−03
41.81	1.683−02	−1.468−02	−1.759−02	5.509−04	−3.428−04	−3.977−04	−1.218−05	2.496−02
31.46	1.163−02	−1.107−02	−1.378−02	4.589−04	−3.530−04	−2.369−04	−1.015−05	3.796−02
21.03	7.220−03	−7.379−03	−1.117−02	3.917−04	−3.547−04	−1.242−04	−8.470−06	4.314−02
10.54	3.390−03	−3.667−03	−9.451−03	3.432−04	−3.538−04	−5.114−05	−7.082−06	4.274−02
.00	0.000	0.000	0.000	2.887−04	−3.217−04	0.000	−7.317−03	−2.057+00

Table 4.3.

REFERENCES

1. Frazer, R. A., Duncan, W. J., and Collar, A. R., *Elementary matrices and some applications to dynamics and differential equations*, Cambridge 1938.
2. Zurmühl, R., *Matrizen und ihre technischen Anwendungen*, Berlin 1964.
3. Falkenberg, J. C., *The General Analysis of Shells of Revolution as applied to Hyperbolic Cooling Towers under the action of Wind Forces and Dead Loads*, University of Southampton 1966.
4. Erugin, N. P., *Linear Systems of Ordinary Differential Equations with Periodic and Quasi-Periodic Coefficients*, Academic Press, New York & London 1966.
5. Fox, L., *Numerical Solution of two-point boundary value problems*, Oxford 1957.
6. Midgley, J. E., *Calculation of subdominant solutions of linear differential equations*, J. SIAM Numer. Anal. Vol. 3 No. 1, 1966.
7. Roberts, S. M., and Shipman, J. S., *Continuation in Shooting methods for twopoint boundary value problems*, Journ. Math. Anal. & Appl. 18, 1967.
8. Goodman, T. R., and Lance, G. N., *Numerical Integration of Two-Point Boundary value problems*, Math. Tables and other aids to comput. 10 (1956), p. 82–86.
9. Bellman, R. E., Kagiwada, H. H. and Kalaba, R. E., *Invariant embedding and the numerical integration of Boundary-Value problems for unstable linear systems of ordinary differential equations*, Communications of the ACM February 1967.
10. Conte, S. D., *The numerical solution of linear boundary value problems*, SIAM rev. vol. 8 No. 3, July 1966.
11. Godunov, S., *On the numerical solution of boundary value problems for systems of ordinary linear differential equations*, Uspehi Mat. Nauk. 16 (1961), pp. 171–174.
12. Pipes, L. A., *A perturbation method for the solution of linear matrix differential equations*, J. Franklin Institute, 283,5 May 1967.

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