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TEKNISKE SKRIFTER

NR. 21 N

DIFFERENCE EQUATIONS FOR PHYSICAL AND TECHNICAL PROBLEMS

By

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1. Introduction.

It is, today, possible to find numerical solutions for difference equations which were difficult or impossible to attack a short time ago. Electronic computers usually give the results easily and rapidly, and if, for some reason, it is not convenient to use an electronic computer, the relaxation method gives a comparatively easy way of arriving at the results.

The difference equations, which are therefore increasingly used for the solution of technical and physical problems, are as a rule derived from differential equations. The domain in question is covered by a mesh, and the differential equations for the problem are approximated by difference equations at the nodes.

The differential equations, in their turn, are obtained from the equilibrium conditions for the problem, written as difference equations, by passing to the limit. They are evolved using certain assumptions as to the conditions in the immediate neighbourhood of the point in question, which of course are correct for infinitesimal elements, but which may not always be made when the distances between the nodes considered are finite.

As will be shown, this procedure may lead to incorrect difference equations when the mesh used has irregular points, caused by a not rectangular section, as in fig. 1, and even more so when changes of material occur, as in fig. 2.

The author will show that in order to arrive at correct difference equations, it is in many cases necessary to use the equilibrium equations for the problem as a point of departure, deriving the difference equations directly from these, without the detour over the differential equations. This procedure gives the same results as approximation from the differential equations at regular points, but at irregular points, and where the material constants change, the results may be very different. In the solution of problems from the theory of elasticity, the main purpose of this paper, many difficulties disappear when the equations are built up this way.

Some of the points above have also been made by *Varga* (1), and his results have been used by *Nohel* and *Timlake* (2). Varga derives the difference equations by integrating the differential equation over each homogeneous region near the point considered, transforming the resulting integrals by means of *Green's* theorem, and then approximating the integral equations with the usual difference formulas. In this way, some of the effects of the finite size of the elements may be taken into account. This method can only be used where the Laplacian operator, as the only derivation of the unknown functions, appears in the equations.

2. Application to a problem in heat flow.

The equation $\Delta w = 0$ [1]

for steady state heat flow without sources, is derived from the equilibrium condition that the rates of heat flow to and from any point must be equal, or

$$\sum_1^4 \lambda_n (w_n - w_o) F_n / h_n = 0 \qquad [2]$$

The usual procedure for approximating [1] for a quadratic mesh, see for instance *Southwell* (3), (4), gives

$$\sum_1^4 (w_n - w_o) = 0 \qquad [3]$$

where, as will be seen, both λ, F and h have disappeared when compared to equation [2].

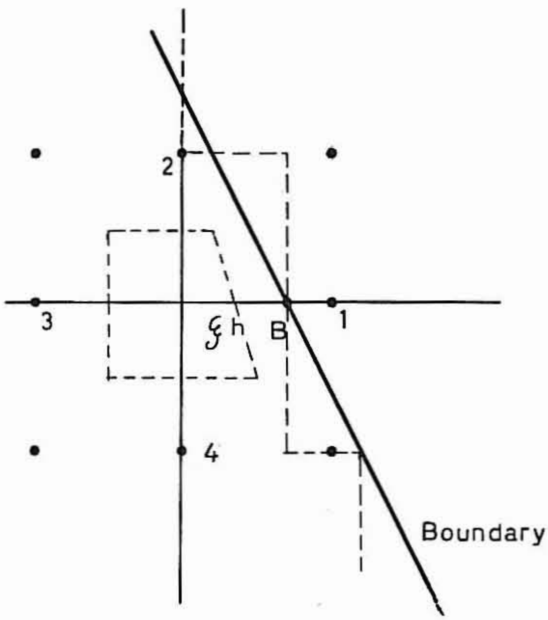

Fig. 1 Mesh with boundary of non-rectangular section.

For an irregular node, fig. 1, such as one frequently finds near the boundary, the finite-difference approximation for $\frac{\partial^2 w}{\partial x^2}$ is (see Allen (5), p. 67, equation (12),

$$h^2 \frac{\partial^2 w}{\partial x^2} = \frac{2 w_B}{\xi(1+\xi)} + \frac{2 w_3}{1+\xi} - \frac{2 w_0}{\xi} \qquad [4]$$

Equation [1] is therefore in this case approximated by

$$\frac{2 w_B}{\xi(1+\xi)} + \frac{2 w_3}{1+\xi} + w_2 + w_4 - \left(2 + \frac{2}{\xi}\right) w_o = 0 \qquad [5]$$

whereas [2] gives

$$\frac{w_B}{\xi h}h + \frac{w_2}{h}\left(\frac{h}{2} + \frac{\xi h}{4}\right) + \frac{w_3}{h}h + \frac{w_4}{h}h - \frac{4 + 10\xi + \xi^2}{4\xi}\, w_o = 0,$$

or

$$\frac{2w_B}{\xi(1 + \xi/2)} + \frac{2w_3}{1 + \xi/2} + w_2 + \frac{2w_4}{1 + \xi/2} - \left(1 + \frac{2}{\xi} \cdot \frac{1}{(1 + \xi/2)} + \frac{4}{1 + \xi/2}\right) w_o = 0 \qquad [6]$$

If, for instance, λ had been different above the line $3 - B$, it would have been complicated to evolve the correct equation from [1], whereas no difficulties appear when using [2].

The difference between [5] and [6] arises, as a comparison with fig. 1 will show, from the fact that in using [1], the boundary must be supposed to run as shown in the dotted line through B. Writing equation [2] for this latter boundary, one arrives at equation [5].

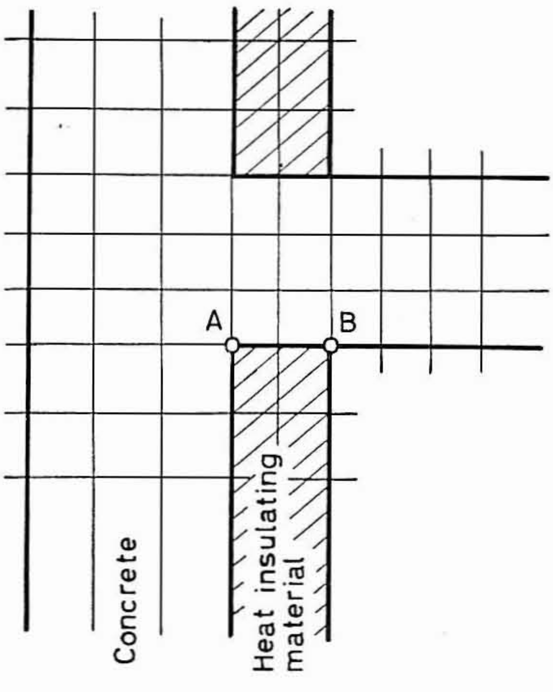

Fig. 2 Mesh with nodes with different coefficient of conduction in different quadrants (Concrete wall and slab with heat insulation).

The difference between equations derived from [1] and [2] may be important in many cases, not only when conditions as in fig. 1 prevail, but especially for problems such as that shown in fig. 2, the calculation of heat loss from a slab supported on an exterior wall. In this case, as will be seen, one finds inevitably points with different heat conductivity in different quadrants, as at A and B, and it is difficult to arrive at the correct equations starting from [1].

Heat transfer problems may perhaps be said to be the least complicated of the problems that are usually solved by the help of difference equations. For more

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complicated problems, such as one encounters for instance in the theory of elasticity, it is, as will be shown, still more necessary to use the equilibrium conditions as a point of departure, in order to arrive at correct difference equations at the boundary.

3. Application to the theory of elasticity.

As an important illustration of the foregoing, equations that can be used for solving two-dimensional problems in the theory of elasticity are developed. In this case, both the equilibrium equations and the equations of continuity for the elements must be written. These will be developed for a state of plane strain; it is a simple matter to go from these to the equations for plane stress when desired. Both can be used for solution of many interesting problems.

The equilibrium equations in the x-direction for the element „0" are (fig. 3).

$$(\sigma_1 - \sigma_3)b_o + (\tau_2 - \tau_4)\, a_o = 0 \qquad [7a]$$

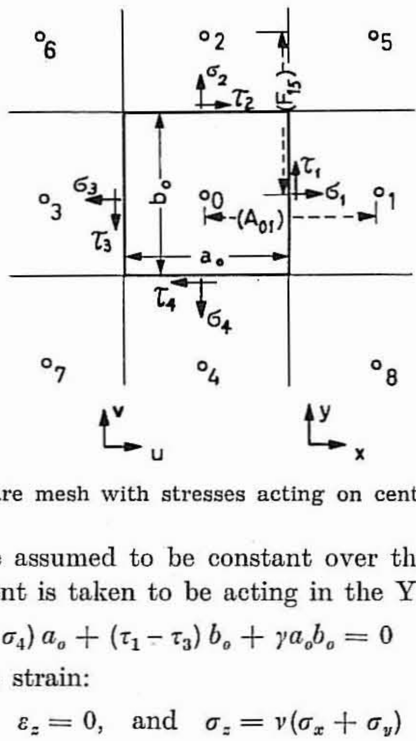

Fig. 3 Square mesh with stresses acting on central element.

The stresses are here assumed to be constant over the length of the element. The weight of the element is taken to be acting in the Y-direction. One finds

$$(\sigma_2 - \sigma_4)\, a_o + (\tau_1 - \tau_3)\, b_o + \gamma a_o b_o = 0 \qquad [7b]$$

For a state of plane strain:

$$\varepsilon_z = 0, \quad \text{and} \quad \sigma_z = \nu(\sigma_x + \sigma_y)$$

and by *Hooke's* law

$$\frac{\delta u}{\delta x} = \varepsilon_x = [\sigma_x - \nu(\sigma_y + \nu\sigma_x + \nu\sigma_y)]/E = [(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y]/E$$

and

$$\frac{\delta v}{\delta y} = \varepsilon_y = [(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x]/E$$

with $\varrho = E/[(1-\nu^2)(1-\nu)-\nu^2(1+\nu)]$, one finds

$$\sigma_x = \varrho\left[(1-\nu)\frac{\delta u}{\delta x} + \nu\frac{\delta v}{\delta y}\right]$$

$$\sigma_y = \varrho\left[\nu\frac{\delta u}{\delta x} + (1-\nu)\frac{\delta v}{\delta y}\right] \text{, and}$$

$$\tau = \frac{E}{2(1+\nu)}\left(\frac{\delta v}{\delta x} + \frac{\delta u}{\delta y}\right)$$

The expressions for σ_x and σ_y given above can now be written as finite differences

$$\sigma_1 = \varrho\,\{(1-\nu)(u_1-u_o)/A_{o1} + \nu[(v_2-v_4)/2 + (v_5-v_8)/2]/(F_{15}+F_{18})\} \qquad [8a]$$

$$\sigma_2 = \varrho\,\{(1-\nu)(v_2-v_o)/A_{o2} + \nu[(u_5-u_6)/2 + (u_1-u_3)/2]/(F_{25}+F_{26})\} \qquad [8b]$$

with the corresponding expressions for σ_3 and σ_4.

The evolution of these equations follows easily from what is said above, by consideration of fig. 3. We have, when the vertical boundary between elements ,,0'' and ,,1'' is considered:

$$\delta u/\delta x = (u_1-u_o)/A_{o1}$$

the difference being taken between the centers of the elements. Further, for the same vertical boundary line:

$$\delta v/\delta y = [(v_2-v_4)/2 + (v_5-v_8)/2]/(F_{15}+F_{18})$$

the difference along the boundary line is here taken as the average of the differences v_2—v_4 and v_5—v_8. The equation [8a] is of course written for the boundary between elements 0 and 1, [8b] for 0—2.

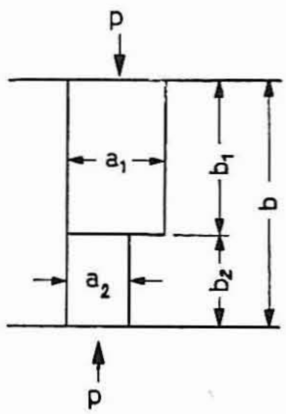

Fig. 4 Deformation of elements with varying dimensions.

A_{mn} and F_{mn} can be found from fig. 4, as follows:

$$\delta = \frac{P}{E}\left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right) = \frac{du}{dx}\,b,$$

$$\text{and for } a_1 = a_2 \qquad \left(\frac{du}{dx}\right)_1 = \frac{\mathrm{P}}{E\,(b_1 + b_2)},$$

therefore

$$\frac{du}{dx} = \frac{\delta}{b_1 + b_2} = \frac{\mathrm{P}}{E}\;\frac{b_1/a_1 + b_2/a_2}{b_1 + b_2} = \left(\frac{du}{dx}\right)_1 \left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right) = \frac{\delta u}{A}\,.$$

$$A = \frac{b_1 + b_2}{b_1/a_1 + b_2/a_2},$$

and therefore

$$A_{o1} = \frac{a_o + a_1}{2} \Big/ \left(\frac{a_o}{2b_o} + \frac{a_1}{2b_1}\right)$$

$$F_{15} = \frac{b_o + b_1 + b_2 + b_5}{4} \Big/ \left(\frac{b_o + b_1}{(a_o + a_1)\,2} + \frac{b_2 + b_5}{(a_2 + a_5)\,2}\right)$$

$$F_{18} = \frac{b_o + b_1 + b_4 + b_8}{4} \Big/ \left(\frac{b_o + b_1}{(a_o + a_1)\,2} + \frac{b_4 + b_8}{(a_4 + a_8)\,2}\right)$$

The ,,A''s are counted between the centers of the elements, the ,,F''s along their boundary lines. The ,,F''s are again, like the differences $\delta v/\delta y$ taken as the averages of the expressions for the neighbouring elements. For the meaning of the indexes, see fig. 3. For inner points, with homogeneous material and all sections of equal size, all A and F are equal to the mesh width a.

Further, for the shear, obtained in the same way as [8a] and [8b]

$$\tau_1 = G\left[\frac{v_1 - v_o}{A_{o1}} + \frac{u_2 - u_4 + u_5 - u_8}{2(F_{15} + F_{18})}\right] \qquad [9a]$$

$$\tau_2 = G\left[\frac{u_2 - u_o}{A_{o2}} + \frac{v_1 - v_3 + v_5 - v_6}{2(F_{25} + F_{26})}\right] \qquad [9b]$$

with the corresponding expressions for τ_3 and τ_4, and $G = E/2(1+\nu)$.

The equations for the moments do not here give the usual result $\tau_1 = \tau_2$, as the normal forces do not necessarily attack at the centerlines of the elements. In order to examine the moment equilibrium in the least complicated way, it is now supposed that $A_{mn} = F_{mn} = a$.

One finds then

$$(\tau_1 - \tau_2) + (\tau_3 - \tau_4) = [(v_1 + v_3 - 2v_o) - (u_2 + u_4 - 2u_o)]\,G +$$
$$+ [(u_5 - u_8 + u_7 - u_6) - (v_5 - v_8 + v_7 - v_6)]\,G$$

It will be seen that the moment is the difference between the second derivatives of u and v at the center, plus the difference between the first derivatives at the sides. This can be cancelled by excentricities of the normal pressures.

Substituting [8] and [9] in [7], gives

$$\begin{aligned}
&\varrho(1-\nu)(u_1-u_o)/A_{o1}+\varrho\nu(v_2-v_4+v_5-v_8)/2(F_{15}+F_{18})\\
&-\varrho(1-\nu)(u_o-u_3)/A_{o3}-\varrho\nu(v_2-v_4+v_6-v_7)/2(F_{36}+F_{37})\\
&+G(u_2-u_o)\frac{a_o}{b_o}/A_{o2}+G(v_1-v_3+v_5-v_6)\frac{a_o}{b_o}/2(F_{25}+F_{26})\\
&-G(u_o-u_4)\frac{a_o}{b_o}/A_{04}-G(v_1-v_3+v_8-v_7)\frac{a_o}{b_o}/2(F_{47}+F_{48})=0
\end{aligned}\qquad [10a]$$

$$\begin{aligned}
&\varrho(1-\nu)(v_2-v_0)/A_{02}+\varrho\nu(u_1-u_3+u_5-u_6)/2(F_{25}+F_{26})\\
&-\varrho(1-\nu)(v_0-v_4)/A_{04}-\varrho\nu(u_1-u_3+u_8-u_7)/2(F_{47}+F_{48})\\
&+G(v_1-v_0)\frac{b_0}{a_0}/A_{01}+G(u_2-u_4+u_5-u_8)\frac{b_0}{a_0}/2(F_{15}+F_{18})\\
&-G(v_0-v_3)\frac{b_0}{a_0}/A_{03}-G(u_2-u_4+u_6-u_7)\frac{b_0}{a_0}/2(F_{36}+F_{37})+\gamma b_0=0
\end{aligned}\qquad [10b]$$

For inner points and homogeneous material, with all $a=b$ and A and F equal to a, [10] gives

$$\begin{aligned}
&\varrho(1-\nu)(u_1+u_3-2u_0)+G(u_2+u_4-2u_0)+\\
&+(\varrho\nu+G)(v_5-v_8-v_6+v_7)/4=0
\end{aligned}\qquad [11a]$$

$$\begin{aligned}
&\varrho(1-\nu)(v_2+v_4-2v_0)+G(v_1+v_3-2v_0)+\\
&+(\varrho\nu+G)(u_5-u_8-u_6+u_7)/4+\gamma a^2=0
\end{aligned}\qquad [11b]$$

The following expressions for the stresses are now obtained

$$\sigma_1=\varrho(1-\nu)(u_1-u_0)+\varrho\nu(v_2+v_5-v_4-v_8)/4 \qquad [12a]$$

$$\sigma_2=\varrho(1-\nu)(v_2-v_0)+\varrho\nu(u_1+u_5-u_3-u_6)/4 \qquad [12b]$$

$$\tau_1=G(v_1-v_0)+G(u_2+u_5-u_4-u_8)/4 \qquad [12c]$$

$$\tau_2=G(u_2-u_0)+G(v_1+v_5-v_3-v_6)/4 \qquad [12d]$$

Equations [11] are the same as would be found taking the differential equations for the plane state of strain, and substituting differences for the differentials, see (6). This checks that the equations [10] are correct. But at the boundary and at all irregular points very different results are to be found, just as in the case of equations [5] and [6].

At free surfaces, the expressions corresponding to σ and τ in the equations [10a] and [10b] must be put equal to zero, or the value of the exterior loading must be introduced instead of these. Thus, for conditions as shown in fig. 5, the part corresponding to τ_2 in [10a] must be made equal to zero, and in [10b] p must be substituted for the part corresponding to σ_2.

In the expressions corresponding to the shears, the differences must be written correspondingly, for instance, fig. 5,

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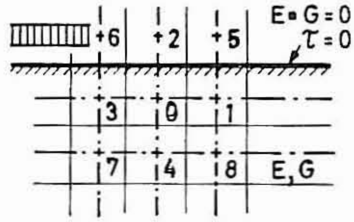

Fig. 5 Boundary conditions for a free boundary with normal load p.

$$\frac{\delta v}{\delta y} = [(v_0 - v_4)/2 + (v_1 - v_8)/2]/F_{18} \qquad [13]$$

For point P in fig. 6, one finds

$$\frac{\delta v}{\delta y} = \left[\frac{v_0 - v_4}{2F_{18}} + \frac{v_5 - v_8}{2(F_{15} + F_{18})}\right] \qquad [14]$$

as before, this is for the vertical boundary between elements 0—1.

Instead of the equations [11], one finds for point zero in fig. 5:

$$\begin{aligned} &\varrho(1-\nu)(u_1 + u_3 - 2u_0) + \varrho\nu(v_1 - v_8 - v_3 + v_7)/2 + \\ &+ G(u_4 - u_0) + G(v_3 - v_1 + v_7 - v_8)/4 = 0 \\ &\varrho(1-\nu)(v_4 - v_0) + \varrho\nu(u_7 - u_8 + u_3 - u_1)/4 + \\ &+ G(u_1 - u_8 - u_3 + u_7)/2 + G(v_1 + v_3 - 2v_0) + pa + \gamma a^2 = 0 \end{aligned} \qquad [15]$$

At fixed boundaries, u and v must, of course, be made equal to zero.

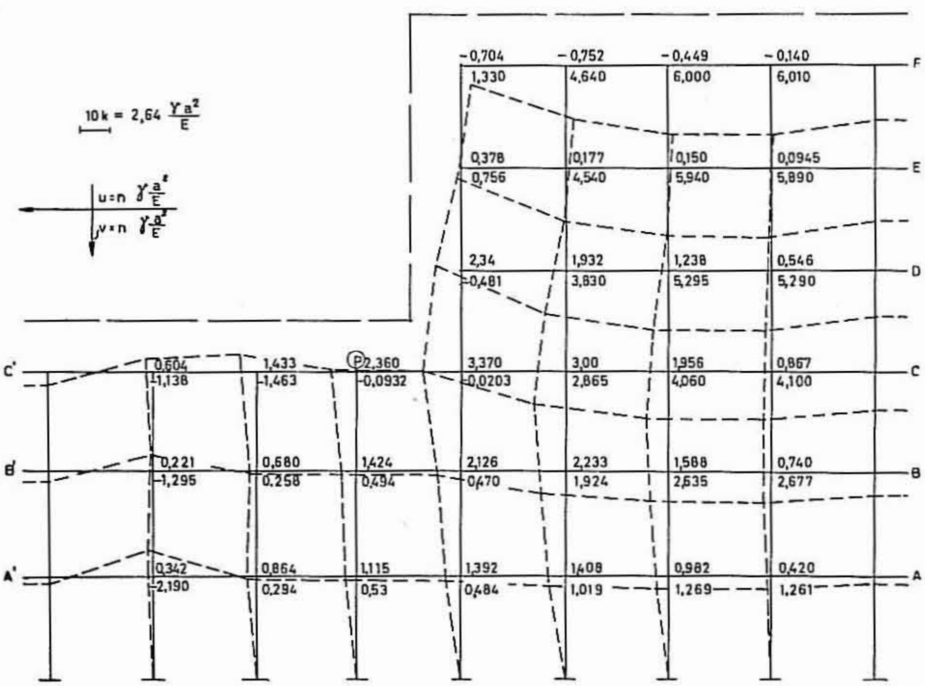

Fig. 6 Section with vertical wall. Deformations from weight of section.

For the stresses, instead of [12a], for surface conditions as in fig. 5, it is found that

$$\sigma_1 = \varrho(1-\nu)(u_1 - u_0) + \varrho\nu(v_0 - v_4 + v_1 - v_8)/2$$

If, as is often the case, the line 3—0—1 had been placed on the boundary, one would have $b_0 = b_1 = b_3 = 0{,}5a_0$ and correspondingly different values for A_{02} etc.

In order, however, to get as small a number of different expressions as possible, it is usually advantageous to locate the last mesh-centerline half a mesh width from the boundary, as shown in fig. 5.

4. Example.

Fig. 6 shows a section with vertical wall, subject to its own weight. The boundary conditions are taken to be $u = v = 0$ at the depth shown in the figure. At a sufficient distance to the right and left of the wall, $v = 0$ and one finds, with $\nu = 0{,}45$

$$\sigma_y = n\gamma a, \qquad \sigma_x = \frac{\nu(1+\nu)}{1-\nu^2}\,\sigma_y = 0.816\,\sigma_y$$

$$E\varepsilon_y = (1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x = 0{,}265\,\sigma_y$$

$$-v_0 = v_4 + \left[\frac{\delta v}{\delta y}\right]_4 \frac{b_4}{2} + \left[\frac{\delta v}{\delta y}\right]_0 \frac{b_0}{2} \qquad \text{(fig. 3)}$$

This gives, with

$$k = \frac{\gamma a^2}{\varrho(1-\nu)}, \qquad \frac{\gamma a^2}{E} = 3{,}79\ k$$

	σ_y	σ_x	$E\,\varepsilon_y$	v
A	5,5 γa	4,49 γa	1,459 γa	$-$ 0,73 γa^2 = $-$ 2,77 k
B	4,5 ,,	3,67 ,,	1,191 ,,	$-$ 2,05 ,, $-$ 7,78 ,,
C	3,5 ,,	2,86 ,,	0,929 ,,	$-$ 3,12 ,, $-$ 11,82 ,,
D	2,5 ,,	2,04 ,,	0,663 ,,	$-$ 3,91 ,, $-$ 14,80 ,,
E	1,5 ,,	1,22 ,,	0,398 ,,	$-$ 4,44 ,, $-$ 16,81 ,,
F	0,5 ,,	0,21 ,,	0,133 ,,	$-$ 4,71 ,, $-$ 17,81 ,,

Corresponding values must be calculated for the points A^1—C^1.

The equations [11] and [15], with variations for instance for point P which will be obvious when considering equations [13] and [14], can now be written for all points. The solution is obtained quickly (and cheaply) from an electronic computer, and is given in fig. 6.

The deformations when the body is subject to its own weight are given here. By subtracting these values in all points from the vertical deformations in a section with horizontal surface, as given in the table above, the deformations caused by an excavation can be found.

If desired, a finer net can be introduced near the corner, and the deformations here can thus be found with the desired exactitude.

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SUMMARY

Difference equations used for solution of physical problems are usually obtained from differential equations, which were originally obtained from equilibrium conditions by passing to the limit. The author shows that this double step frequently leads to incorrect difference equations near the boundary and at other irregular points, and to difficulties in satisfying the boundary conditions. It is shown that it usually is simpler, and frequently more correct, to evolve the difference equations directly from the equilibrium conditions.

A solution of a problem from the theory of elasticity is given as an example.

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